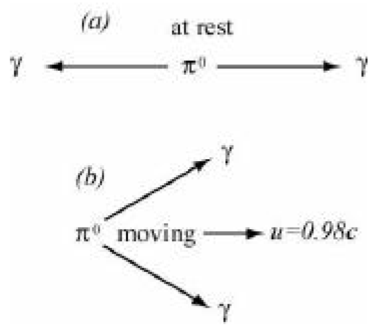
Lecture 10   February 1, 2002

Example: pi meson decay



A pion has a rest energy of 135MeV. It decays into two gamma rays (photons). Consider a pion traveling at *v=0.98c* with respect to the lab frame decays into two gamma rays of equal energy, make equal angle θ with respect to the direction of motion. Find the energy, momentum, and θ of the gamma rays.

From conservation of total energy the energy of the gamma ray in the lab frame is:

$$E_g = \frac{1}{2}E_{p^0} = \frac{1}{2}\times\frac{135MeV}{\sqrt{1-0.98^2}} = 339.2MeV$$

The absolute value of the momentum (which is along the x-axis) of the pion is:

$$p_p = \frac{1}{c}\sqrt{E_p^2-(m_p c^2)^2} = \sqrt{678.4^2-135^2}\,MeV/c = 664.8MeV/c$$

Thus, the x-component of the gamma ray (conservation of momentum):

$$p_{gx} = \frac{1}{2}\times 664.8MeV/c = 332.4MeV/c$$

We also know that

$$p_g = E_g/c = 339.2MeV/c.$$

Therefore, from $p_{gx} = p_g\cos q$, we get

$$\cos q = \frac{332.4MeV/c}{339.2MeV/c} = 0.978 \text{ and } q = 11.5°$$

We can derive the angle θ from a slightly different approach. Let's consider the problem from the point of view of an observer moving with the pion. In that frame, the momentum *p* is zero before the pion decay. Thus, after the pion decay the momentum has to be zero as well in that frame. Therefore, the two gamma rays have to move in the opposite direction in that frame. Since we only consider the case where, after the decay, the two gamma rays move symmetrically with respect to the direction of motion (the x-axis) as observed in the lab frame, the decay in the pion frame has to occur along the y'-axis.

From conservation of total energy the energy of the gamma ray in the pion frame is:

$$E_g = \frac{1}{2}E_{p^0} = \frac{1}{2}\times 135MeV = 67.5MeV$$

Thus, the momentum along the y’-axis is (photon is massless):

$$p_{y'} = \pm E_g / c = \pm 67.5MeV / c\,.$$

Also, decay along the y’ direction leads to

$$p_{x'} = 0$$

We know that $v_g = c$ in all inertial frames. Thus, since $v_x^{'} = 0$, $v_y^{'} = c$

$$v_x = \frac{v_x^{'} + u}{1 + u\times v_x^{'}/c^2} = \frac{0 + 0.98c}{1 + 0.98c\times 0/c^2} = 0.98c$$

$$v_y = \frac{v_y^{'}}{1 + u\times v_x^{'}/c^2}\sqrt{1 - u^2/c^2} = \frac{c}{1 + 0.98c\times 0/c^2}\sqrt{1 - (0.98c)^2/c^2} = 0.199c$$

Thus, the angle θ is

$$\tan q = \frac{0.199c}{0.98c} = 0.203 \text{ and } q = 11.5°$$

Example: fusion

The atomic mass of the ^{4}He atom is 4.002602u. Find the energy released when 2 neutrons and 2 protons combine to form ^{4}He.

$$2m_p c^2 + 2m_n c^2 - M_{^4He}c^2 = (2\times 1.007276u + 2\times 1.008665u - 4.001504u)c^2$$

$$= 0.0304uc^2 = 0.0304uc^2 \times \frac{931.5MeV}{uc^2} = 28.3MeV$$

Example

An electron, initially at rest, is accelerated by a potential V=1.00x10^{6}Volts. After such acceleration, the electron moves at a constant speed *v*. Calculate this speed ($m_e c^2 = 0.511MeV$).

After the acceleration the kinetic energy of the electron is 1.00MeV. Thus, the total energy *E* is 1.00*MeV*+0.51*MeV*=1.51*MeV*. Therefore,

$$pc = \sqrt{E^2 - m_e^2 c^4} = \sqrt{1.51^2 - 0.51^2}\,MeV = 1.42MeV$$

From

$$p = \frac{m_e v}{\sqrt{1 - v^2/c^2}}$$

we get

$$(1.42MeV)^2 = \frac{m_e^2 c^2 v^2}{1 - v^2/c^2} = \frac{m_e^2 c^4 v^2/c^2}{1 - v^2/c^2} \text{ or } (1.42MeV)^2 = \left[(1.42MeV)^2 + m_e^2 c^4\right]\frac{v^2}{c^2}$$

Thus,

$$v = c\sqrt{\frac{(1.42MeV)^2}{(1.42MeV)^2 + (0.511MeV)^2}} = \frac{1.42}{1.509}c = 0.94c = 2.82\times 10^8 m/s$$

Another way of solving this.

$$K = \frac{m_e c^2}{\sqrt{1 - v^2/c^2}} - m_e c^2 = 1MeV$$

Thus,

$$\sqrt{1 - v^2/c^2} = \frac{0.511MeV}{1MeV + 0.511MeV}$$

This gives:

$$v = 0.941c = 2.82 \times 10^8 m/s$$

Example

A space station is moving horizontally (along the x-axis) at a speed *u*=0.6c with respect to earth. A rocket of 1 kg is launched vertically (y-axis) from the earth at v_y=0.5*c*. (a) What is the kinetic energy of the rocket according to the astronaut on the space station? (b) What is the momentum of the rocket along the x'-axis (of the space station parallel to the x-axis) according to the astronaut?

(a) $v_x^{'} = \frac{v_x - u}{1 - (u/c^2)v_x}$. Thus, $v_x^{'} = \frac{0 - 0.6c}{1 - (u/c^2) \times 0} = -0.6c$

$v_y^{'} = \frac{v_y}{1 - (u/c^2)v_x}\sqrt{1 - u^2/c^2} = \frac{0.5c}{1 - (u/c^2) \times 0}\sqrt{1 - 0.36} = 0.4c$. Thus,

$v' = \sqrt{(v_x^{'})^2 + (v_y^{'})^2} = \sqrt{0.36 + 0.16}c = 0.721c$

Therefore,

$$K' = \frac{mc^2}{\sqrt{1 - v'^2/c^2}} - mc^2 = \left(\frac{1}{\sqrt{1 - v'^2/c^2}} - 1\right)mc^2$$

$$= 0.443 \times \left(3 \times 10^8\right)^2 kg \bullet m^2 \bullet s^{-2} = 3.99 \times 10^{16} J$$

(b) Therefore, $p_x^{'} = \frac{mv_x^{'}}{\sqrt{1 - (v^{'})^2/c^2}} = \frac{-1.0kg \times 0.6 \times 3 \times 10^8 m/s}{\sqrt{1 - 0.721^2}} = -2.5 \times 10^8 kg \bullet m/s$