Below are the UNC Department of Physics and Astronomy Qualifying exams from the years 1999 through 2012. Due to changes in the exam format and poor record keeping, some of the sections are ambiguously labeled or missing. Numbers in red are best guesses about the exact exam. Years with missing problems are labeled by how many are included in this packet. This packet starts with the 2012 exam and moves backwards.

1999 CM, EM1	May 2000 CM, SM, QM1, EM1
Spring 2001 CM, SM, EM1, QM1	Winter 2001 EM2, QM2
Spring 2002 CM, SM, EM1, QM1	Winter 2002 EM1, QM1, Stars, HE
Spring 2003 CM, SM, EM1, QM1	Winter 2003 EM2, QM2, HE
Winter 2004 EM2, QM2, Stars, HE	Spring 2004 CM, SM, EM1, QM1
Spring 2005 CM, SM, EM1, QM1	Winter 2005 EM2, QM2
2006 CM, SM, EM1, QM1, EM2, QM2, HE (1/5)	2007 CM, SM, EM1, QM1, EM2, QM2, Stars, HE
2008 CM (4/5), SM, QM2 (3/5), EM2 (2/5), Stars (3/5), Galaxies (3/5)	2009 CM, SM, EM1, QM1, EM2, QM2, Stars, Galaxies
2010 CM, SM, EM1, QM1, EM2, QM2, Stars, HE	2011 CM, SM, EM1, QM1, EM2, QM2, Stars, Galaxies
2012 CM, SM, EM1, QM1, EM2, QM2, Stars, HE	

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part I: Classical mechanics and Statistical mechanics

Saturday, May 12, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

CM: Classical Mechanics
Work out 3 out of 5 problems

SM: Statistical Mechanics
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part II: Electromagnetism I and Quantum mechanics I

Monday, May 14, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMI: Electromagnetism I
Work out 3 out of 5 problems

QMI: Quantum Mechanics I
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part III: Electromagnetism II and Quantum mechanics II

Monday, May 14, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMII: Electromagnetism II
Work out 3 out of 5 problems

SM: Quantum mechanics II
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2012

Part III: Astro I and II

Monday, May 14, 2012

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)__________________

Astro I:
Work out 3 out of 5 problems

Astro II:
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature________________________

Print name___________________________

CM-1

A point particle of mass m moves in a plane in response to the following Lagrangian

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\alpha\dot{x}\dot{y}\right) - \frac{1}{2}k\left(x^2 + y^2 + 2\beta xy\right),$$

where $k > 0$ is a spring constant and α and β are two other time-independent parameters.

(a) Find the normal mode frequencies, $\omega_{1,2}$.

(b) What conditions must be placed on α and β so that the motion is always a bounded, stable oscillation?

(c) Find the eigenvectors of the system.

(d) Sketch the motion in the x and y coordinate system for the two eignemodes. How does the behavior depend upon the values of α and β?

CM-2

Consider a straight cylindrical shaft of radius $\alpha \approx 1\,\text{m}$ that penetrates through the center of the earth. It emerges at latitudes θ and $-\theta$ relative to the equator. Assume that the earth is a perfectly homogenous sphere of mass M and radius R that rotates with angular velocity ω. A test point mass of mass m is suspended at rest relative to the shaft directly above the symmetry axis of the shaft and then released to free-fall down the shaft. You may neglect air-friction and relativistic effects for this problem. Answer the following questions:

(a) Use a non-inertial Cartesian coordinate system where one axis points down the center of the shaft and another along the east-west direction to provide a *qualitative* description of the test mass' motion.

(b) Find the equations of motion of the test mass in the reference frame from part (a). Recall that the fictitious force experienced by a mass m in a reference frame rotating at a constant angular velocity is given by:

$$\mathbf{F} = \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{centrifugal}} = -2m\mathbf{\Omega} \times \mathbf{v_r} - m\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

(c) Discuss the motion in the specific cases where $\theta = 0°$ and $\theta = 90°$. If the test mass experiences any oscillatory motion, find the period of the oscillation(s).

(d) In the "shallow approximation", α is small enough so that the test mass only falls a short distance, $d \ll R$, before it hits the side of the shaft. Find an algebraic equation for d in the shallow approximation. Do not attempt to solve it.

(e) Assume that the test mass is dropped down the hole in the northern hemisphere. If you stand over the shaft so that north is at 0 radians, at what angle will the test mass hit the side of the shaft in the shallow approximation?

CM-3

A particle of mass m moves under the influence of gravity, with downward acceleration g, on the inner surface of the paraboloid of revolution, $x^2 + y^2 = az$. Here $a > 0$ is a length scale and the surface is assumed to be frictionless. The coordinate z points upward in the vertical direction.

(a) Write down the Lagrangian and show that the equations of motion can have the form

$$\ddot{\rho} - \rho\dot{\phi}^2 = 2\lambda\rho,$$
$$d(\rho^2\dot{\phi})/dt = 0,$$
$$\ddot{z} = -g - a\,\lambda,$$

and

$$a\,z = \rho^2,$$

where we have converted to cylindrical coordinates (ρ, ϕ, z).

(b) Now let the particle be given an angular velocity of $\sqrt{2g/a}$, so that it orbits in a horizontal circle. Prove the stability of the particle in this circular path by showing that if the particle is displaced slightly from this path, while holding the angular momentum fixed, it will undergo oscillations about the path.

(c) Find the frequency of the radial oscillations.

CM-4

Consider the Lagrangian for a two-dimensional system

$$L = \dot{q}_1\dot{q}_2 - cq_1q_2\,,$$

where c is a positive constant.

(a) Solve the equations of motion and describe the physical system that the Lagrangian defines.

(b) The Lagrangian is invariant under the scale transformation

$$q1 \longrightarrow e^{\lambda}q_1\,, \qquad q_2 \longrightarrow e^{-\lambda}q_2\,,$$

for arbitrary λ. Use Noether's theorem to find the conserved quantity associated with this invariance, and interpret its meaning.

(c) Suppose the two coordinates q_1 and q_2 are the x and y coordinates of a single object. What property of the object does the conserved quantity describe?

CM-5

A ball is bouncing vertically and perfectly elastically in an elevator that accelerates from rest with acceleration $a(t)$. The rate of change of the acceleration $a(t)$ is very slow: $\dot{a}(t)T \ll g$, where T is the period of the ball's motion and g is the usual acceleration of objects in the earth's gravitational field.

(a) The equivalence principle says that this situation is equivalent to what other situation?

(b) If the ball has maximum height h_0 above the floor of the elevator before the acceleration begins, what is the maximum height $h(t)$ at a later time t, in the adiabatic limit?

SM-1

Consider an ideal Bose gas (non-relativistic) confined to a region of area A in two dimensions. Express the number of particles in the excited states, N_ε, and the number of particles in the ground state, N_0, in terms of z, T, and A, and show that the system does not exhibit Bose-Einstein condensation unless $T \to 0$ K.

$\left(g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x - 1};\quad 0 \le z \le 1 \quad \text{and } g_n(1) = \zeta(n) \text{ and } \zeta(1) = \infty \right)$

SM-2

Consider N identical, localized, noninteracting spins with spin quantum number j. The magnetic moment of each spin is $\vec{\mu} = g\mu_B\vec{j}$ where μ_B is the Bohr magneton, g is the Lande factor, and the eigenvalues of j_z, the magnetic quantum number m, are $m = -j, -j+1, \ldots j-1, j$. In the presence of an external magnetic field $\vec{H}$ the energy of the system is given by $E = -\sum_{i=1}^{N} \vec{\mu}_i \cdot \vec{H}$.

(1) Show that the magnetization is given by $M_z = Ng\mu_B jB_j(x)$

where $x = g\mu_B Hj / k_B T$ and $B_j(x)$ is the Brillouin function given by

$$B_j(x) = (1+\frac{1}{2j})\coth\left((1+\frac{1}{2j})x\right) - \frac{1}{2j}\coth\left(\frac{x}{2j}\right)$$

(2) Show that for $x = g\mu_B Hj / k_B T \ll 1$, the magnetic susceptibility is given by $\chi = N\frac{g^2\mu_B^2 j(j+1)}{3k_B T}$

$\left(\coth x \approx x^{-1} + \frac{1}{3}x \text{ for } x \ll 1\right)$

SM-3

Consider a classical gas of N identical particles. The energy of the system is given by

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m} + \sum_{i<k} U_{ik}(|\vec{r}_i - \vec{r}_k|)$$

In the dilute (atomic volume $\times N \ll V/N$) and high temperature ($|U| \ll k_B T$) approximation it can be shown that the partition function can be written as

$$Z(T,V,N) = \frac{1}{N!}\left(\frac{1}{\lambda^3}\right)^N Q_N(V,T) \quad \text{where} \quad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

where the configurational integral $Q_N(V,T)$ is given by

$$Q_N(V,T) = V^N + V^{N-2}\sum_{i<k}\int d^3r_i \int d^3r_k \left(e^{-U_{ik}/k_BT} - 1\right)$$

Assume the potential is given by the hard sphere potential

$$U_{ik}(|\vec{r}_i - \vec{r}_k|) = \begin{matrix} +\infty & |\vec{r}_i - \vec{r}_k| < r_0 \\ 0 & |\vec{r}_i - \vec{r}_k| \ge r_0 \end{matrix}$$

Show that the equation of state is given by

$$P\left(V - N\frac{2\pi}{3}r_0^3\right) = Nk_BT$$

SM-4

Consider the closed (this means that the nearest neighbors of spin 1 are spin 2 and spin N) 1 D Ising model where the Hamiltonian is given by

$$H_N(\sigma_1,\ldots,\sigma_N) = -I\sum_{n.n.}\sigma_i\sigma_k - \mu B\sum_{i=1}^{N}\sigma_i \; ; \quad \sigma_i = \pm 1 \text{ and } n.n.\text{=nearest neighbors}$$

μ is the magnetic moment, B is the magnetic field, and I is the coupling strength. The mean-field approximation predicts a spontaneous magnetization below a critical temperature $T_c = 2I/k_B$. The exact solution shows that the mean-field approximation made qualitatively wrong prediction. Such phase transition does not exist in 1D. Show that the exact free energy is given by

$$F = -k_BT\ln\left(\lambda_1^N + \lambda_2^N\right)$$

where

$$\lambda_{1,2} = e^{I/k_BT}\cosh\left(\mu B/k_BT\right) \pm \sqrt{e^{-2I/k_BT} + e^{2I/k_BT}\sinh^2\left(\mu B/k_BT\right)}$$

From this you can derive the magnetization and see why there is no spontaneous magnetization at T>0 K. However, you don't have to show this here.

SM-5

A cylinder of radius R and length L contains N molecules of mass m of an ideal gas at temperature T. The cylinder rotates about its axis with an angular velocity ω. Find a change in the free energy of the gas ΔF, as compared to that at rest.

EMI.1

(a) For an arbitrarily moving charge, the charge and current densities are $\rho(\vec{r},t) = e\delta(\vec{r}-\vec{R}(t)), \vec{j}(\vec{r},t) = e(d\vec{R}/dt)\delta(\vec{r}-\vec{R}(t))$, where $\vec{R}(t)$ is the position of the charged particle. Verify the statement of conservation of charge.

(b) Find the total charge and the electric dipole moment of the charge density $\rho(\vec{r}) = -\vec{d}\cdot\nabla\delta(\vec{r})$.

(c) What electromagnetic fields do the following potentials describe? $\phi = 0, \vec{A} = \vec{a}(\vec{a}\cdot\vec{r})$, where $\vec{a}$ is a constant vector.

EMI.2

Start with the potential due to a given local charge distribution around the origin of the coordinate system $\phi(\vec{r}) = \int(d\vec{r'})\frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|}$. If the total charge of the given charge distribution is zero, show that (a) the potential , in its leading behavior for large distances, has the form

$\phi(\vec{r}) = \frac{\vec{r}\cdot\vec{d}}{r^3}$ where the electric dipole moment is given by $\vec{d} = \int(d\vec{r'})\vec{r'}\rho(\vec{r'})$.

(b) Consider an additional point charge e_1 located at a point $\vec{r}$ lying far from the dipole; the interaction energy is given by $E = \vec{d}\cdot\frac{e_1\vec{r}}{r^3}$. Show that we can interpret this energy as the interaction energy of the dipole moment with the electric field $\vec{E}$ produced by e_1 at the origin, i.e., $E = -\vec{d}\cdot\vec{E}$.

(c) Use $E = -\vec{d_1}\cdot\vec{E}$ as the interaction energy of an electric diple moment $\vec{d_1}$ with the field $\vec{E}$ produced by a given charge distribution far from $\vec{d_1}$. Calculate the interaction energy E for dipole-dople interaction, i.e., for the interaction of $\vec{d_1}$ at the origin with the field $\vec{E}$ produced by another dipole moment $\vec{d_2}$ located at $\vec{r}$.

EMI.3

(a) Show that a perfectly conducting sphere of radius a placed in a constant magnetic field $\vec{B}_0$ acquires a magnetic moment $\vec{\mu} = -\frac{1}{2}a^3\vec{B}_0$.

(b) Find the surface current density $\vec{K}$.

(c) Show that the values for $\vec{\mu}$ and $\vec{K}$ are consistent with each other.

EMI-4

Consider two straight parallel wires, carrying static charge with linear charge density of ρ and $-\rho$, respectively. The wires are along the z-direction, one is located at $x = {}^{a}/_{2}$ and the other at $x = -{}^{a}/_{2}$.

(a) Find the electric potential and the electric field everywhere in space.

(b) Simplify your expression for the region far away from the wire, and express the field in terms of the linear dipole density $p=\rho a$.

EMI-5

Consider two parallel plane electrodes (regarded as infinite) separated by a distance d. The cathode located at $x = 0$ with electric potential of $\varphi(x = 0) = 0$ is capable if emitting unlimited electrons (charge *e* and mass *m*) when an electric field is applied to it. The electrons leaving the cathode with zero initial velocity are accelerated toward the anode located at $x = d$ with electric potential of $(x = d) = V_0$. In the steady state there will be a constant electric current flowing from the cathode to the anode.

(a) Find a relationship between the current density J, the space charge density $\rho(x)$ and the electric potential $\varphi(x)$ in the space between the two electrodes. Is *J* a constant or a function of *x*, *why?*

(b) Derive a differential equation that determines the electric potential $\varphi(x)$.

(c) Assuming a power law solution ($\varphi(x)$ is proportional to x^p), solve for the potential density *J* in terms of *e*, *m*, *d*, and V_0.

QM1-1 Two electrons interact via a spin-spin interaction that is given as $\alpha\, \mathbf{S_1} \cdot \mathbf{S_2}$ where α is a constant. *One* of the electrons is also trapped in a region with a homogeneous external magnetic field of intensity B_0. Please answer the following questions:

(a) What is the sign of α? Explain your answer.

(b) Consider *only* spin degrees of freedom and find the allowed energies of this system in terms of fundamental constants and α.

(c) Assume that we create an ensemble of these two electron systems. For each member of this ensemble, we perform a measurement of the spin of the electron that is trapped in the magnetic field region along the direction of the magnetic field. What is the average energy of the subset of the ensemble with measured spin in the same direction as the magnetic field?

QM I-2

(a) Consider a Hamiltonian $H(\lambda)$ that depends on a parameter λ, one of the Hamiltonian's eigenstates $|\varphi(\lambda)\rangle$, and the corresponding energy $E(\lambda)$. Show that

$$\frac{dE(\lambda)}{d\lambda} = \langle\varphi(\lambda)|\, \frac{dH(\lambda)}{d\lambda}\, |\varphi(\lambda)\rangle \ ,$$

a result that is known as the Hellman-Feynman theorem.

(b) The states of the hydrogen atom with no radial nodes $(n = l + 1)$ have energies

$$E(l) = \frac{-e^4 m}{2\hbar^2 (l+1)^2}$$

Letting $H(\lambda)$ be the Hamiltonian for the radial Schrödinger equation and the parameter λ be l, use the Hellman-Feynman theorem to derive an expression for the expectation value of the operator r^{-2} in states with no radial nodes.

QM1-2 Consider an electron that is free to jump between 3 fixed, identical atoms in a molecule. Each atom is located at a corner of an equilateral triangle. Ignore spin and any other nearby electrons and atoms. We can define an orthornormal basis set of states of the electron to be spherically symmetrical orbitals bound to each atom. In other words, $|S_i\rangle$ would correspond to an electron bound to the ith atom. In this basis the Hamiltonian for the system has all off-diagonal elements equal to ϵ and the diagonal elements equal to zero. Please answer the following questions:

(a) Given that one of the energy eigenvalues is 2ϵ, find the other energy eigenvalues.

(b) The Hamiltonian is obviously invariant under rotations of $2\pi/3$. Find a matrix to represent such a rotation operator (R) and find the simultaneous eigenstates of H and R. You may find the fact that $R^3 = 1$ useful.

(d) Assume that at $t = 0$ the electron is in the $|S_1\rangle$ state. Find the probability that the electron will stay bound to that electron as a function of time.

QMI-4

A paradox:

(a) Show that for finite-dimensional matrices A and B,

$$\mathrm{Tr}[A, B] = 0 .$$

(b) In the 1-d harmonic oscillator, the raising and lowering operators $a^\dagger$ and a obey the commutation relation

$$\left[a, a^\dagger\right] = I .$$

The trace of I is obviously not zero. In the basis of oscillator eigenstates, write down the matrix representations of a and $a^\dagger$ (you can write down the upper left parts and indicate the rest with dots). Multiply them together to get the matrix representations of $aa^\dagger$ and $a^\dagger a$ and explain why the result from part a) doesn't apply.

QMI-5

(a) Write down or derive the equation of motion for the density operator $\hat{\rho}(t)$.

(b) Use the solution of (a) to show whether or not a mixed state can evolve into a pure state.

(c) The *reduced density operator* for a two-particle system is defined as the trace of the two-paticle density operator over the states of the second particle. For the simple two-particle density operator $\rho = |\alpha_1\alpha_2\rangle\langle\beta_1\beta_2|$, the reduced density operator is given by

$$\rho_R = |\alpha_1\rangle\langle\beta_1| \; \mathrm{Tr}\left[|\alpha_2\rangle\langle\beta_2|\right] .$$

All two-particle density operators can be written as sums of simple ones like that above, and ρ_R is defined for these more complicated cases in the obvious way, by invoking linearity.

The reduced density operator is an effective operator for a single particle that takes into account our complete ignorance of the other particle. Consider a two-spin density operator associated with the pure spin-singlet state, i.e.

$$\rho = |S=0\rangle\langle S=0|$$

where

$$|S=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) .$$

Find the reduced density operator. Does it correspond to a pure single-particle state or a mixture of single-particle states? If the former, what state, if the latter, what polarization?

EMII-1

A nonrelativistic particle of mass m and charge e, and initial kinetic energy E, makes a head-on collision with a fixed central force region with potential energy $V(r)$. The particle comes from an infinite distance away. The potential energy steadily increases toward the center so that

$$V(r) < E, \qquad \text{for} \qquad r > r_0,$$

$$V(r) > E, \qquad \text{for} \qquad r < r_0.$$

1. Find the instantaneous total radiated power as a function of position.
2. Integrate the power over all time to find an expression for the total radiated energy.
3. Assuming the potential energy and its derivative at the turning point are finite, show that the integrated emission is finite.

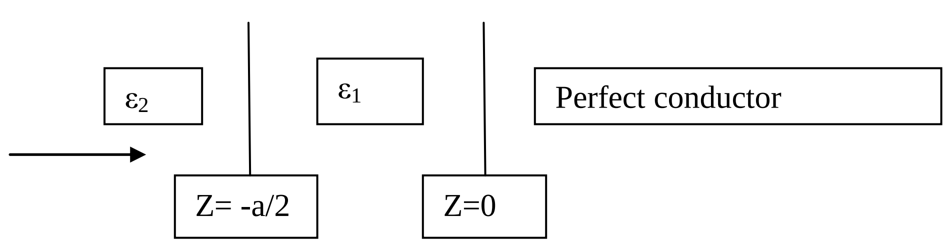
EMII-2

A high energy photon of energy E encounters an electron of mass m and charge e at rest. A scattering occurs with a photon of energy E' emerging and moving at an angle θ relative to the direction of motion of the original photon. The electron recoils with some Lorentz factor γ and angle ψ.

1. Find the expression for the scattered photon's energy in terms of the original photon's energy E, the electron mass m, and the scattering angle θ.
2. Derive an expression for the Lorentz factor $\gamma = E_e^{\text{recoil}}/(mc^2)$ of the recoiling electron.

EMII-3

Two parallel dielectric media are backed by a perfect electric conductor as shown in the accompanying figure. A source, to the left of the first interface, initiates an incident plane wave described by $e^{ik_2(z+a/2)}$ with perpendicular polarization (i.e., the electric field being perpendicular to the plane of incidence.) The reflected wave is represented in terms of $r_a e^{-ik_2(z+a/2)}$. Find the reflection coefficient r_a. Next, find the absolute value of the coefficient and then give a brief physical interpretation of the result.



EMII-4

Start with the Maxwell's equations in vacuum in terms of the electric field $\vec{E}$, the magnetic field $\vec{B}$, the charge density ρ and the current density $\vec{j}$. Write $\vec{E}$ and $\vec{B}$ in terms of scalar potential ϕ and vector potential $\vec{A}$.

(a) Show that, in the Lorenz gauge, the potentials obey the differential equations: $-D\vec{A} = 4\pi\vec{j}/c$ and a similar equation for ϕ, where D stands for the d'Alembertian.

(b) Show that the potentials can be solved in terms of the sources $\vec{j}$ and ρ by using the Green's function technique with the Green's function obeying the equation $-DG(\vec{r}-\vec{r'}, t-t') = 4\pi\delta(\vec{r}-\vec{r'})\delta(t-t')$.

(c) Assume that the equation has been solved (you do NOT have to solve the equation) to yield $G(\vec{r}-\vec{r'}, t-t') = \frac{1}{|\vec{r}-\vec{r'}|}\delta(\pm\frac{1}{c}|\vec{r}-\vec{r'}| - (t-t'))$. Which sign should we use and why? Finally write down the potentials as integral equations of the sources.

EMII-5

Consider a coaxial waveguide. Let the inner radius be b, the outer radius a. Assume $a - b << a$. Find the cutoff wavenumbers for the TM mode. [Hint: You do not need Bessel functions and you may note that for $\Phi(\rho)$ satisfying $(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{m^2}{\rho^2} + \gamma^2)\Phi(\rho) = 0$, one has $(\frac{d^2}{d\rho^2} - \frac{m^2}{\rho^2} + \frac{1}{4}\frac{1}{\rho^2} + \gamma^2)\sqrt{\rho}\Phi(\rho) = 0$.]

QMII-1

a) Assuming that the hamiltonian is invariant under time-reversal, prove that the wavefunction for a spinless non-degenerate system at any given instant of time can always be chosen to be real.

b) The wavefunction for a plane wave state at t=0 is given by a complex function $\mathbf{e}^{\mathbf{i\,px}}$. Why does this not violate time reversal invariance?

QMII-2

A p-orbital electron characterized by |n, l=1, m = + 1, -1, 0> (ignore spin) is subjected to a potential $\mathbf{V = \lambda\,(x^2 - y^2)}$ where $\boldsymbol{\lambda}$ =constant.

a) Obtain the 'correct' zero-order energy eigenstates that diagonalize the perturbation. You don't need to evaluate the energy shifts in detail, but show that the original 3-fold degeneracy is now completely removed.
b) Because V is invariant under time reversal and because there is no longer any degeneracy, we expect each of the energy eigenstates obtained in (a) to go into itself, (up to a phase factor) under time-reversal. Check this point explicitly.

QMII-3

Work out the quadratic Zeeman effect for the ground state of hydrogen atom, $[\langle x|0\rangle = 1 / (\pi a_0^3)^{1/2}\, e^{-r/a_0}\,]$ due to the neglected term " $\mathbf{e^2 A^2 / 2m_e c^2}$ " in the hamiltonian taken to first order. Write the energy shift as $\boldsymbol{\Delta} = -\chi\, B^2/2$ and obtain an expression for " χ " .

This is a useful integral**:** $\int_0^\infty e^{-ar}\, r^n\, dr = n! /(a^{n+1})$.

QMII-4

Three spin -0 particles are situated at the corners of an equilateral triangle. Let us define the z-axis to go through the center and in the direction normal to the plane of the triangle. The whole system is free to rotate about the z-axis. Using statistics considerations, obtain restrictions on the magnetic quantum numbers corresponding to $\mathbf{J_z}$.

QMII-5

Consider scattering from the delta-shell potential $V(r) = g\,\delta\,(r - r_0)$.

a) First determine the boundary conditions at $r = 0$ and $r = r_0$, then make a suitable ansatz, apply the boundary conditions, and compute the s-wave scattering amplitude.
b) Determine the s-wave bound states of an infinite spherical well of radius r_0. Comment on the relation of the delta –barrier resonance and these bound states. What happens to the s-wave scattering length when the incident *k*-value sweeps across the "*k*" corresponding to one of these quasi bound state

AstroI-1 White Dwarfs

a) The pressure integral: $$P = \frac{1}{3}\int_0^\infty pvn(p)dp$$

allows you to calculate pressure given a distribution in momentum *n(p)dp*. Assuming that a completely degenerate electron gas has the electrons packed as tightly as possible, so that their separation is of order $n_e^{-1/3}$, use the Heisenberg uncertainty principle to estimate the momentum of an electron in terms of n_e. By further assuming that p = m_ev (non-relativistivic) and that all the electrons have the same momentum (to make the integral trivial), derive the exponent η in the power law equation of state:

$$P \propto \rho^\eta T^0$$

(For this problem, don't worry about the constants of proportionality, they'll be wrong under the constant momentum assumption anyway.)

b) Now use your understanding of this equation of state and hydrostatic equilibrium and mass conservation in scaling law form to plot white dwarf cooling curves on a log Teff,- log L (H-R) diagram. Work in solar units and use the normalization that an 0.6 solar mass white dwarf has a radius of 0.01 Rsun at solar Teff. Plot curves for 0.2, 0.6, 0.8 1md 1.0 solar mass white dwarfs.

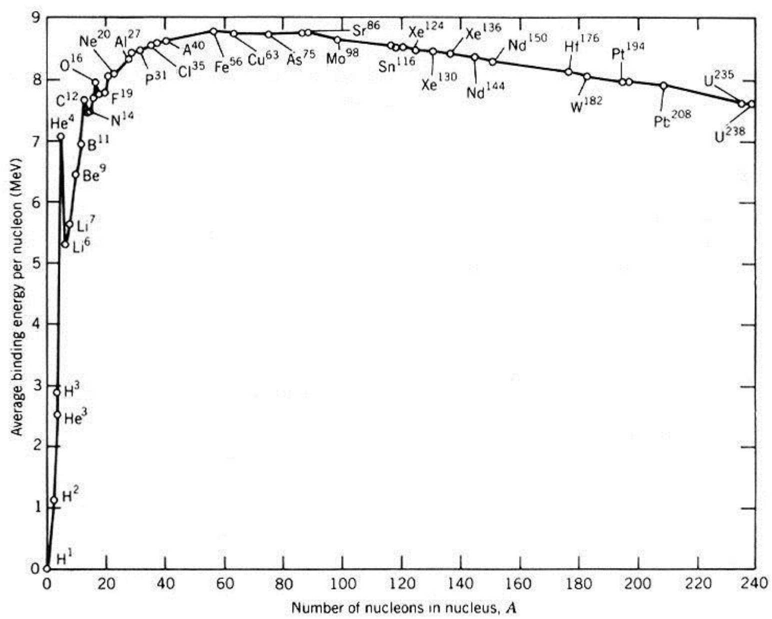
c) Now write the mass-luminosity relationship for non-relativistic white dwarfs in power law form.

AstroI-2 Deuterium burning in stars

a) In the formation of a main sequence star from a protostar there is a phase in which primordial Deuterium is fused. This happens at a temperature of 10^6 degrees rather than the 15 x 10^6 required for P-P reactions. Use hydrostatic equilibrium and mass conservation, along with the ideal gas law, to compare the radius of a 1 solar mass protostar in its D burning phase to its radius on the main sequence. You need to assume the density profiles of the two stars are identical (one can be scaled to the other).

b) Referring to the curve of binding energy below, estimate the total energy available from D burning for a solar mass star if the primordial D abundance is 0.013% of P, and occurs in the inner 10% of the star. (Deuterium fuses via 1H + 2H –> 3He + γ)

c) Compare the D-burning timescale to the Kelvin Helmholtz (gravitational contraction) timescale.

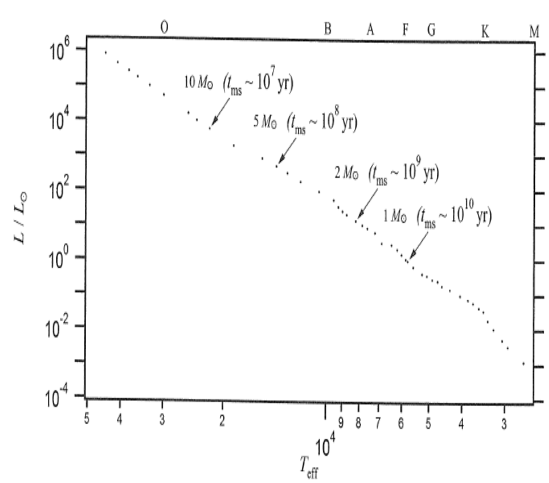


AstroI-3 Lifetime, luminosity, mass scaling

a) Use the plot below to estimate the exponent for a power law relation for main sequence lifetime in terms of stellar mass: $L \propto M^{\alpha}$

b) Assuming that all main sequence stars convert the same fraction of their total mass to He, what is the expected Mass-Luminosity relation on the main sequence? How closely do the luminosities expected from this relation match the luminosities seen in the H-R diagram depicted? Comment.

c) Assuming the gas in these stars is ideal (a fairly good assumption) and that the central temperature is proportional to the effective temperature (not so good), use the assumption of hydrostatic equilibrium and the Mass-Luminosity relation from above to estimate the temperature dependence of the nuclear reactions (assume negligible density dependence).



Astro I-4

α Centauri A is G2 like the Sun but older, having 1.14 $M_\odot$, 1.23 $R_\odot$, and 1.5 $L_\odot$. It is 1.34 parsecs from us in an 80-year elliptical orbit with α Centauri B. B is type K1, has 0.92 $M_\odot$, 0.86 $R_\odot$, 0.5 $L_\odot$, and approaches A to 11.2 AU separation. Assume that both stars emit as blackbodies and consider a planet in a circular orbit around B.

(a) Assume that the planet cools as a blackbody through a non-greenhouse gas atmosphere that reaches 1 Earth surface pressure. Calculate the inner (water steam formation) orbital radius in AU of the "habitable" zone around B, ignoring for now star A. Take planet albedo as 50% and assume rapid rotation.

(b) Show quantitatively that there is no significant change in the zone's outer radius (water ice formation) around B even when star A is closest.

(c) Like Earthlings, the α Centaurians are loading CO_2 into their atmosphere as they rapidly burn up fossil fuels. Assuming that their planet has average temperature 40 °C but that the increasing CO_2 will soon push it to 50 °C, calculate

i. what fraction of the sunlight should be blocked by a perfectly opaque sunscreen to reduce insolation on the planet to restore its pre-CO_2 equilibrium temperature, or alternatively,

ii. by how much they must increase planetary albedo at visible wavelengths through chemical modification.

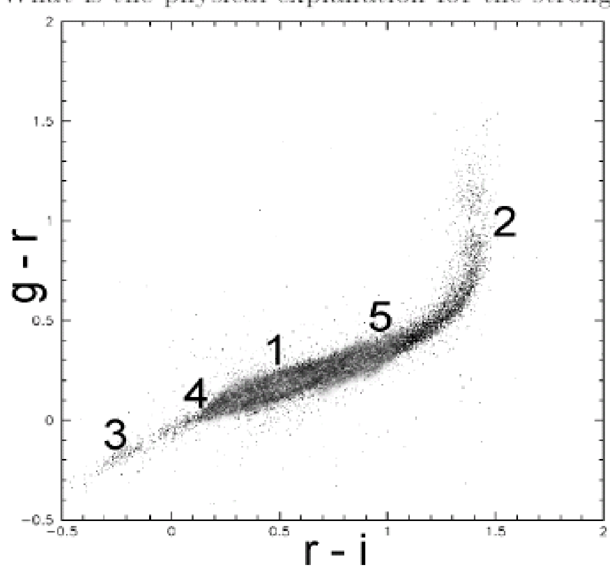
AstroI-5

The diagram below plots colors of many stars with Sloan r < 17 through the 4 SDSS filters indicated, over a field in our Galaxy. Assume that the colors have been corrected for reddening.

(a) Where feasible, identify the spectral classes, evolutionary stages, and approximate masses of stars near each of the numbered regions.

(b) Assuming that most of these stars formed in a burst, estimate the mean age today of this population if they show similar chemical composition to that of the Sun.

(c) What is the physical explanation for the strongly curved "tail" at 2?



AstroII-1

It is possible to calculate the nuclear statistical equilibrium at high densities between neutrons, protons, and electrons in a neutron star by treating each as an ideal Fermi-Dirac gas component. For equilibrium to occur there has to be a balance between

$$n \to p + e^- + \bar{\nu}_e,$$

and

$$p + e^- \to n + \nu_e,$$

and in a neutron star we assume the neutrinos escape.
At high enough densities the muon (another fermion) can appear, changing an ideal $n - p - e$ gas into an ideal $n - p - e - \mu$ gas. If it is energetically feasible, the two reactions,

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e,$$

and

$$e^- \to \mu^- + \bar{\nu}_\mu + \nu_e,$$

may occur.

(a) Write down the set of thermodynamic equilibrium equations between dimensionless Fermi momenta x_e, x_p, x_n, and x_μ, and the expression for the total energy density.

(b) In terms of the masses of the particles, m_e, m_p, m_n, and m_μ, determine x_e, x_p, and x_n at threshold for the appearance of muons in the gas. (You do not need to obtain numerical values.)

AstroII-2

The Universe contains cosmic ray particles, including a very high energy power-law distribution. The very highest energy cosmic ray protons are measured to have energies up to around 10^{20} eV. At that energy the spectrum cuts off, though there is controversy over the statistics of the very highest energy events.

The GZK mechanism is thought to limit the highest energy that a proton can have because scattering of the relativistic proton off of a cosmic microwave background photon can, at a certain threshold, produce a pion. Once threshold for pion production is reached, the proton loses approximately 20% of its energy per pion scattering. The reaction is

$$p + \gamma_{\rm cmb} \rightarrow p' + \pi^0,$$

where the most favorable case is for the CMB photon to be traveling in the opposite direction of the initial proton.

(a) Let the proton mass be m_p, the pion mass be m_π, and the CMB photon energy be E. The Lorentz factor of the proton before scattering is γ and after scattering is γ'. Assume the reaction is just at threshold to make a pion. Find the expression for the required initial proton Lorentz factor γ in terms of the other masses and energies.

(b) Take the proton mass to be $m_p = 938$ MeV, the pion mass to be $m_\pi = 135$ MeV, and the CMB photon energy to be $E = 2.5 \times 10^{-4}$ eV (note: eV). Find the approximate GZK cutoff energy for the protons.

ASTR II Problem 3

A thin accretion disk surrounds a Schwarzschild black hole. The gas can be treated as if approximately in isolated circular orbits. Recall that (more general) radial orbital motion satisfies

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - V(r),$$

where the effective potential $V(r)$ is

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right).$$

Here $\tilde{E}$ is the relativistic specific energy ($\tilde{E} \to 1$ for particles just unbound at infinity) and $\tilde{L}$ is the specific angular momentum (units with $G = c = 1$).

(a) Go through the calculation and show that the innermost stable circular orbit (ISCO) is at $r = 6M$.

(b) Find the values of $\tilde{E}$ and $\tilde{L}$ at the ISCO.

Assume that $\xi = 1 - \tilde{E}$ is the radiative efficiency of the disk. Assume further that at any given time the disk is being fed with mass at just the right rate $\dot{M}$ to maintain Eddington luminosity, $dE/dt = L_{\rm edd}$.

(c) Material at the ISCO plunges into the black hole via a short spiral. Assume the ISCO orbital constants are preserved during this brief plunge. Derive an equation for the growth of the black hole mass (in terms of the various physical constants **including** G and c). Solve for $M(t)$ assuming that $M = M_0$ at $t = 0$.

(d) Estimate how long it takes for the black hole to spin up to $a/M = J/M^2 \simeq 0.9$.

In parts (c) and (d) continue to treat the black hole as if it remains a Schwarzschild black hole.

ASTROII-4. Binary Survival in Supernova Explosion and Kick Velocity

a. A progenitor of a supernova of mass M_p and a companion star of mass M_c are in a circular orbit about each other of semi-major axis $a = a_p + a_c$, where a_p and a_c are with respect to the system's center of mass. Determine expressions for a_p/a and a_c/a.

b. Determine an expression for the angular speed $\omega_p = \omega_c = \omega$ of the stars an a function of $M = M_p + M_c$ and a.

c. Determine expressions for the velocities v_p, v_c, and $v = v_p + v_c$ of the stars as functions of M_p, M_c, M, and a.

d. The progenitor supernovas leaving behind a neutron star of mass M_{NS}. Determine an expression for the minimum mass that the companion star must have for the binary to survive (as a function of M_p and M_{NS}).

e. Determine an expression for the momentum of the supernova shell.

f. Determine an expression for the kick velocity of the binary system.

ASTROII-5. Colonization of the Galaxy

Assuming that humanity has mastered efficient, controlled fusion as an energy source, estimate how long it will take for us to colonize the Galaxy. (The text is long, but the calculations are short.)

a. Assume that the mass of our unfueled ships is similar to the mass of our fuel supply (^{3}He, collected from gas giants at each stop). Also assume that we are generating energy via ^{3}He + ^{3}He $\rightarrow$ ^{4}He + 2p and converting it to kinetic energy near 100% efficiency (difficult to do, but theoretically possible). Very roughly, estimate how fast our ships would go. (The simplest back-of-the-envelope estimate matches the exact calculation within a factor of ~2, so do not waste time on the exact calculation.)

$m_p = 1.007276$ u
$m_{^3He} = 3.016029$ u
$m_{^4He} = 4.002602$ u

b. How long would it take our ships to travel from our location to the far side of the Galaxy if not interrupted by stopping to colonize worlds?

c. Roughly, how many star systems are in the Galaxy? Roughly, what is the volume of the Galaxy in cubic light years? Consequently, what is the typical distance between star systems? How long would it take our ships to travel this distance?

d. Assume that all stars have at least one planet in the traditional habitable zone. But also assume that humanity is not interested in tidally-locked planets. If the atmosphere is thin, only the ring around the planet in constant twilight would be habitable (a far way to travel for not much surface area). If the atmosphere is thick, it will redistribute the heat from the star-facing side to the dark side and the entire surface would be habitable, but the winds could be violent.

The distance at which planets tidally lock to their stars scales with $M^{1/3}$, where M is the mass of the star. In our solar system, Mercury is tidally locked but Venus is not. The distance at which planets are in the traditional habitable zone scales with $L^{1/2}$, where L is the luminosity of the star. In our solar system, Earth is in the middle of the traditional habitable zone. Assuming a reasonable stellar mass-luminosity relation, estimate below what stellar mass M roughly Earth-mass/size planets near the middle of the traditional habitable zone are tidally locked.

e. Very roughly, what fraction of stars have masses above M? Also very roughly, of these what fraction of stars are not in binary (or multiple) systems? (In binary systems, planets can only be in stable orbits if close enough to one of the stars to be tidally locked, or if far enough from both stars to be outside of the traditional habitable zone.) Given these factors, and any others that you wish to include, estimate what fraction of stars have planets acceptable for colonization (assuming that the remaining stars have at least one planet, or moon around a hot Jupiter, in the traditional habitable zone that can be sufficiently terraformed; we will not consider worlds in non-traditional habitable zones, such as tidally-heated moons of regular, cold Jupiters).

f. Given your estimate, what is the typical distance between habitable planets? How long would it take our ships to travel this distance?

g. Generational ships must be large enough to support a population with sufficient genetic diversity, but not so large of a population as to make the ship too expensive/time-consuming to build and fuel. Assume 100 - 1000 people per ship. Given humanity's current level of technological and medical advancement, our population is doubling every 40 years. Arguments can be made for both faster and slower growth rates, with future levels of technological and medical advancement, but slowed by the challenges of terraforming and/or (presumably controlled) genetic adaptation to the new environment. Simply using humanity's current rate, roughly estimate how long it would take a generational crew to fully populate an Earth-like planet?

h. Assume that we fully populate each world before we send out new generational ships, presumably in all (unsettled) directions. The time that it would take to colonize the entire Galaxy is then no different than the time that it would take to colonize an approximately direct path from our location to the far side of the Galaxy (of course avoiding the Galactic center). Given your estimates, how long is this time?

Is this timescale short or long? Specifically, if this timescale is much shorter than the typical timescale for a habitable planet to develop life and civilization to the point of mastering fusion, the first civilization to master fusion will likely get the whole Galaxy. Given that we are probably only hundreds of years away from mastering fusion (at most), in this case the Galaxy is either 100% ours, or it has already been fully colonized by an earlier civilization which for whatever reason decided to leave us alone (but in which case the Galaxy is 0% ours). If this timescale is much longer than the typical timescale for a habitable planet to develop life and civilization to the point of mastering fusion, the Galaxy will likely be colonized by many civilizations, in which case we are likely to get at least a part of it, but only a part. Assuming that humanity survives the next few hundred years, what does our long-term future look like?

Useful facts:

$F = Gm_1m_2/d^2$

$G=4.3 \times 10^{-6}$ kpc $(km/s)^2/M_{sun}$

1kpc = 1Gyr x 1km/s

Virial Theorem 2KE+PE=0

Poisson Equation $\nabla^2\varphi = -4\pi G\rho$

Wien's Law T = 3 mm·K / λ_{peak}

centripetal force $F=mV^2/r$ (uniform circular motion)

Faber-Jackson Relation $L \propto \sigma^4$

Dynamical time $t_{dyn}=\sqrt{2}t_{ff}$

Crossing time t_{cross} = R/V = 1 Gyr (**R** in kpc/**V** in km/s)

Relaxation time $t_{relax} = 0.1N / \ln(N)\ t_{cross} = 10^6$ yr x 0.1N/ln(N) x (**R** in pc/**V** in km/s)

Numerical Constants:

Solar Mass (Msun):	1.989×10^{33} g
Solar Radius (Rsun):	6.96×10^{10} cm
Solar Luminosity:	3.847×10^{33} erg/s
Gravitational Constant (G)	6.6726×10^{-8} $cm^3/g/s^2$
Proton mass	1.6726×10^{-24} g = 938.27 MeV/c^2
Yield of p-p reactions (Q)	26.7 MeV = 4.28×10^{-5} ergs
Boltzmann constant	1.38×10^{-16} erg/K
Planck's Constant	6.626×10^{-27} erg-s
Electron mass	9.109×10^{-28} g

$m_p = 1.6726 \times 10^{-24}$ g

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part I: Classical mechanics and Statistical mechanics

Friday, May 7, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

CM: Classical Mechanics
Work out 3 out of 5 problems

SM: Statistical Mechanics
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part II: Electromagnetism I and Quantum mechanics I

Monday, May 9, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMI: Electromagnetism I
Work out 3 out of 5 problems

QMI: Quantum Mechanics I
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part III: Electromagnetism II and Quantum mechanics II

Monday, May 9, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)_________________

EMII: Electromagnetism II
Work out 3 out of 5 problems

SM: Quantum mechanics II
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Doctoral Written Examination in Physics, 2011

Part III: Astro I and II

Monday, May 9, 2011

Instructions: Please work in the assigned room, but take a break outside anytime you want to. Mathematical handbooks and electronic calculators are allowed.
Begin each answer on a new sheet and write only on one side of each sheet.
Identify each sheet by:

Page___________ of Question___________Student's # (PID)________________

Astro I:
Work out 3 out of 5 problems

Astro II:
Work out 3 out of 5 problems

(Partial credit will be given for partial answers)

My work is completed in full observance of the Honor code:

Signature_________________________

Print name____________________________

CM-1

A point particle of mass $\boldsymbol{m}$ and charge $\boldsymbol{q}$ is attached to the end of a massless pendulum of length $\boldsymbol{l}$. The motion of the pendulum is confined to a plane. Let the pivot of the pendulum be fixed at a height $\boldsymbol{h}$ above an infinite horizontal conducting surface, with $\boldsymbol{h} > \boldsymbol{l}$. Ignore gravity in considering the motion of the pendulum.

1. Use an angular coordinate and obtain the Lagrangian.
2. Find the frequency of small amplitude motion.

CM-2

A particle of unit mass moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = z$, which is assumed to be frictionless. (z is the vertical direction.)

1. Obtain the equations of motion in cylindrical coordinates. (You do not have to solve them.)
2. What angular momentum must be given to the particle so that it describes a horizontal circle at the height $z = 1$?

CM-3

Consider a satellite in circular Earth orbit and the stability of its orientation relative to the Earth. Let the mass of the Earth be $\boldsymbol{M}$ and the radius of the orbit be r_0. By Kepler's third law the angular frequency of the orbit satisfies

$\Omega^2 = GM/r_0^3$.

The satellite is an extended rigid body that can be idealized as two masses, $\boldsymbol{m}$, separated by a massless rigid rod of length $2a$. The rod lies initially in the orbital plane and you should consider only motion in that plane. The satellite may rotate in the plane and therefore could corotate with its orbit, maintaining a fixed orientation with respect to the Earth.

1. Let the angle between the rigid rod and the direction to the Earth be given by Ψ. Write down the kinetic energy of motion relative to the circular orbit of the center of mass (i.e., you are to take the center of mass motion as known).
2. Write down the potential energy of the rigid body and expand it in powers of a/r_0 to find the leading non-vanishing Ψ dependent terms.
3. Write down the Lagrangian and obtain the equation of motion for satellite orientation (relative to the Earth; i.e., for motion in Ψ).
4. Via an effective potential or other means find the equilibrium orientation angles. Which of these equilibria are stable and which are unstable?

CM-4

Consider a particle under the influence of a force that is constant in space but grows linearly with time. The corresponding Hamiltonian is

$$H = \frac{p^2}{2m} - \lambda x t\,,$$

where λ is a constant. Use the Hamilton-Jacobi method to find $q(t)$ for initial conditions $q(0) = q_0$ and $p(0) = p_0$.

Hint: You can separate variables if you add a term to the generating function to cancel the term containing xt. Try $S = S_x(x) + S_t(t) + \frac{1}{2}\lambda x t^2$ (where any dependence on the integration constant α is not shown). And you should be able to tell whether your answer makes sense.

CM-5

Determining motion by Taylor expansion and Poisson brackets:

1. Show for any function $A(q,p)$ and a time-independent Hamiltonian H that
$$\underbrace{[\,[\cdots[A,H],H],\cdots,H]}_{ntimes} = \frac{d^n A}{dt^n}\,.$$

2. Show that
$$q(t) = q(0)+[q,H]\Big|_{t=0} t+\frac{1}{2}[\,[q,H],H]\Big|_{t=0} t^2++\frac{1}{6}[\,[\,[q,H],H],H]\Big|_{t=0} t^3+\cdots\,,$$
as long as the series converges.

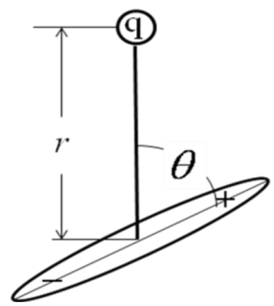
3. Without solving any differential equations, use the results from part (2) to obtain the solution $q(t)$ for a simple harmonic oscillator with spring constant k, mass m, and initial conditions $q(0) = q_0$, $p(0) = p_0$.

SM-1

The phonon modes of a crystal are treated as 3N independent harmonic oscillators. The associated energy is given by $E\{n_i\} = \sum_{i=1}^{3N}(n_i + 1/2)\hbar\omega_i$ and $n_i = 0,1,2\ldots$, and the distribution function of modes in angular frequency is given by $g(\omega)$ where $\int_0^\infty g(\omega)\,d\omega = 3N$. Show that the entropy associated with the phonons is given by $S = \frac{1}{T}\int_0^\infty \frac{\hbar\omega}{e^{\hbar\omega/kT}-1} g(\omega)\,d\omega - k\int_0^\infty \ln\left(1-e^{-\hbar\omega/kT}\right) g(\omega)\,d\omega$

SM-2

Compare two situations: (1) a charge q interacts with an electric dipole $\vec{p}$ that has a fixed orientation and is at a distance r, and (2) a charge q interacts with a dipole $\vec{p}$ that orients freely over all possible angles at a distance r. For case (1) the potential energy is given by $u(r)=\frac{qp\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2}$ where ε_0 is the permittivity and θ is shown below. Show that the interaction in case (2) is shorter-ranged than case (1) by showing that in case (2) $u(r)=-\frac{1}{3kT}\left(\frac{qp}{4\pi\varepsilon_0}\right)^2\frac{1}{r^4}$. Here, assume that |u(r)|<<kT, so that $e^{-u/kT}\approx 1-u/kT$.



SM-3

The partition functions of N particle (of mass m) classical ideal gas contained in a volume V and at temperature T is given by $Q_N(V,T)=\frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N$ where $\lambda=h/\sqrt{2\pi mkT}$. Show that the partition function of an ideal Fermi gas of two particles is $Q_N(V,T)=\frac{1}{2!}\left(\frac{V}{\lambda^3}\right)^2\left(1-\frac{1}{2^{3/2}}\frac{\lambda^3}{V}\right)$. Useful integral: $\int_0^\infty x^2e^{-\alpha x^2}dx=\frac{1}{2\alpha^{3/2}}\frac{1}{2}\sqrt{\pi}$.

SM-4

A particle of mass m with momentum (p_x,p_y,p_z) and coordinates (x,y,z) moves freely in a volume V. (a) Find the normalized distribution function $f(p_x)$ of the x-component of the momentum according to the classical micro-canonical ensemble with energy E.
(b) The corresponding canonical distribution (with temperature chosen to give averaged energy =E) is quite different from the micro-canonical distribution (you do NOT have to show this). Why do you think the two distributions are so different?

SM-5

A long vertical tube with a cross-section area ***A*** contains a mixture of ***n*** different ideal gases, each with the same number of particles **N**, but of different masses $\boldsymbol{m_k}$, ***k=1,…n***. Find a vertical position of the center of mass of this system in the presence of the Earth's gravity, assuming a constant altitude-independent free fall acceleration ***g***.

EMI-1

Consider a very long solenoid with radius *R, N* turns per unit length, and current *I*. Coaxial with the solenoid are two long cylindrical shells of length *l* - one, inside the solenoid at radius *a*, carries a charge +*Q* uniformly distributed over its surface; the other, outside the solenoid at radius *b*, carries charge –*Q*; *l* is supposed to be much greater than $b > R > a$. (Assume there is an electric field only in the region between the cylinders.) When the current in the solenoid is gradually reduced, the cylinders begin to rotate.

a) What are the torques on the outer cylinder and on the inner cylinder?

b) After the current is switched off, how much angular momentum have the two cylinders picked up?

c) Before the current is reduced, what is the total angular momentum in the fields?

EMI-2

A constant charge per unit length $\lambda = dQ/dz$ is distributed along an infinite-length insulator of negligible cross section. A charge ***q*** is present in the vicinity of the line charge.

1. Find the electric field and electric potential due to the line charge.
2. Determine how much work ***W*** is done on the point charge ***q*** if its distance from the line charge increases from cylindrical distance $R = a$ to distance $R = b$.
3. As $b \to \infty$, what happens to the work on the particle?
4. If the line charge were truncated to a finite total length of ***L*** (with ends at $z = \pm L/2$), give an approximate expression for the total work ***W*** done if the charge is taken radially away from $z=0$ and $R=a$ to $R=\infty$.

EMI-3

Calculate the interacting force between a dipole moment $\vec{p} = p\,\hat{z}$ and a conducting sphere of radius ***a***. The dipole moment is at a distance ***R (>a)*** away from the center of the sphere.

EMI-4

A charged sphere with radius *a* is placed in a media with dielectric constant ε. The charge distribution inside the sphere is given by $\rho(x, y, z) = \rho_0 \left(\frac{2z^2 - x^2 - y^2}{a^2}\right)$.

a) Show that the electric scalar potential along the *z*-axis outside the sphere is given by $V(x = 0, y = 0, z) = \frac{\rho_0}{35\varepsilon_0} \frac{a^5}{z^3}$.

b) Using the result in a) to find the general expression of the scalar potential $V(x, y, z)$ everywhere outside the sphere.

EMI-5

A spherical shell of permeability $\boldsymbol{\mu}$ is placed in a uniform field $\boldsymbol{B_o}$. If the internal and external radius of the shell are ***a*** and ***b***, respectively.

a) Find the magnetic field in the hollow interior. Be sure your solution reduces to the obvious result when ***a* =*b*** *(the shell is gone).*

b) Show that in the limit of large permeability the field is of order $\boldsymbol{B_o/\mu}$, thus this shell can act as a magnetic shield.

QM1-1

A spin-1/2 particle with magnetic moment $\boldsymbol{\mu}$ is in an eigenstate of $\boldsymbol{S_x}$ with eigenvalue ***h/2*** at time ***t = 0***. At that time it is placed in a magnetic field of magnitude ***B*** pointing in the z-direction and allowed to precess for time ***T***. At that time the magnetic field is rotated very, very rapidly, so that it now points in the y-direction. After another time interval ***T***, $\boldsymbol{S_x}$ is measured. What is the probability that it is found to be ***h/2***?

QMI-2

Four electrons are each localized to separate atoms in a crystal. The atoms are located at the corners of a regular tetrahedron, which is a triangular pyramid where each face is an equilateral triangle. The length of each edge is ***a***. Find the correction to the energy levels of the four electron system due to the spin-spin interaction between the electrons. You may assume that the spin-spin interaction term between any two electrons ***i*** and ***j*** is of the form:

$$H_{ij} = A\frac{\mathrm{S_i} \cdot \mathrm{S_j}}{r_{ij}^3}$$

where ***A*** is a constant and $\boldsymbol{r_{ij}}$ is the distance between the two electrons. You may also assume that ***a*** is much larger than the spatial extents of the electrons' wave functions, in other words the electrons are distinguishable by their atom's locations on the crystal lattice.

QMI-3

Consider a particle of mass ***m*** trapped in a one-dimensional simple harmonic oscillator well with a resonance angular frequency **ω**
Let |z> be a normalized eigenstate with eigenvalue ***z*** of the raising operator ***a***:

$$a = \frac{1}{\sqrt{2m}}(p + im\omega x)$$

Note that a is not a hermitian operator, hence z can be a complex number.
(a)Show that |z> satisfies the following relationship:

$$\sigma_x = m\omega\sigma_p$$

(b) Show that |z> satisfies the minimum uncertainty relationship between ***x*** and ***p***. You may find the result from part (a) useful for this part.
Comment: |z> is known as a coherent state and has many interesting properties.

QMI-4

Consider three distinguishable particles with spin 1/2 (and no spatial degrees of freedom).
(a) What are the possible values for the total angular momentum of all three particles? Are there any values that have more than one multiplet associated with them?
(b) Write explicit expressions for all the states in the basis that has definite values of the total angular momentum and z-projection.

QMI-5

Consider the one-dimensional Schroedinger equation with

$$V(x) = \begin{cases} \frac{m}{2}\omega^2 x^2 & \text{for} \quad x > 0, \\ +\infty & \text{for} \quad x < 0. \end{cases}$$

Find the energy eigenvalues.

EMII-1

Consider a pulse of electromagnetic radiation in vacuum. In a region within the midst of the pulse, the integrated momentum is

$$\vec{P} = \frac{1}{4\pi c} \int d^3x \, (\vec{E} \times \vec{B}),$$

and the integrated energy is

$$\mathcal{E} = \frac{1}{8\pi} \int d^3x \, (E^2 + B^2).$$

Assume that the following relationship holds between the integrated momentum and energy of the pulse:

$$|\vec{P}| \, c = \mathcal{E},$$

just like that of a single photon. From this relation (for the volume integrals) alone, show that $\vec{E} \cdot \vec{B} = 0$ and $E^2 = B^2$.

One of the crucial steps involves proving the inequality

$$(\vec{E} \times \vec{B})^2 \leq \frac{1}{4} (E^2 + B^2)^2.$$

EMII-2

Find the total cross section σ for the scattering of an electromagnetic wave of long reduced wavelength $\lambda/2\pi$ by a dielectric sphere of radius a (with $\lambda \gg a$) and permittivity ϵ.

You may recall that a dielectric sphere acquires a dipole moment $\vec{d}$ in a static electric field $\vec{E}$ given by

$$\vec{d} = \frac{\epsilon - 1}{\epsilon + 2} a^3 \, \vec{E}.$$

EMII-3

Two relativistic electrons with the same Lorentz factor γ approach each other obliquely. The particles have equal but opposite angles, $\pm\theta$, relative to the x axis. The electrons have just sufficient energy to create a π^+-π^- pair. After the collision the original two electrons emerge along with the newly created pair:

$$e^- + e^- \to e^- + e^- + \pi^- + \pi^+.$$

1. Determine how the Lorentz factor γ depends upon angle θ and the particle masses, m_e and m_π, if the reaction is just at threshold.

2. Determine the Lorentz factor γ' of the particles exiting the event.

3. Calculate γ for the two special cases: $\theta = 0$ and $\theta = \pi/2$.

EMII-4

The spectral-angular distribution of radiation from a relativistic electron is given by

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left| \int dt'\, \vec{n} \times (\vec{n} \times \vec{\beta}(t')) \exp[i\omega(t' - \vec{n}\cdot\vec{x}(t')/c)] \right|^2 .$$

A perfect conductor fills the region $x > 0$. Empty space exists in the region $x < 0$. An observer is situated in empty space and confined to the x-y plane. The associated outward-directed unit vector is $\vec{n} = (\cos\theta, \sin\theta, 0)$. For such an observer, $\pi/2 < \theta < \pi$.

A relativistic electron $(-e)$, with velocity $\vec{\beta} = (\beta, 0, 0)$ in the vacuum region, approaches the conducting surface. The electron strikes the surface at the origin of coordinates at $t' = 0$ and abruptly disappears. This event causes transition radiation.

1. Since the electron was moving relativistically toward the conductor, and radiation will not propagate inside the conductor, explain why transition radiation can even exist in this problem.

2. Derive the spectral and angular distribution of the radiation that can be seen by the observer.

EMII-5

A circularly polarized light wave emerges from a laser and is propagating in vacuum. The light is nearly a plane wave directed along the z axis with wave number k. However, because the beam has finite transverse extent, the amplitude of the electric field drops off (gradually) in the x and y directions. The electric field can be approximately represented by

$$\vec{E} \simeq \frac{1}{\sqrt{2}}\,(\vec{e}_1 + i\vec{e}_2)\,E_0(x,y)\,e^{ikz}\,e^{-i\omega t} \;+\; a(x,y)\,\vec{e}_3\,e^{ikz}\,e^{-i\omega t},$$

where

$$\frac{1}{E_0}\left|\vec{\nabla}E_0(x,y)\right| \ll k.$$

The latter implies that the length scale for changes in the field in the x and y directions is many wavelengths $\lambda = 2\pi/k$ long.

1. Explain why the electric field must have a non-vanishing z component.
2. Show how the amplitude $a(x,y)$ depends upon $E_0(x,y)$.
3. Why is this only an approximate expression for the electric field?

QMII-1

A particle of mass m is trapped in a 2-dimensional infinite potential well with sides of length ***a***. The well is centered on the origin and the edges are parallel to the coordinate axes. The particle also experiences a "Gaussian wall" potential perturbation given as:

$$V'(x,y) = Ae^{-x^2/b^2}$$

where ***b*** << ***a*** and ***A*** is a constant with appropriate units. Use first order perturbation theory to find the ground and first excited state energy levels and their degeneracies. Approximate any integrals that you cannot evaluate easily. You may find the following integral useful:

$$\int_{-\infty}^{\infty} e^{-x^2/b^2}\,dx = b\sqrt{\pi}$$

QMII-2

In the interaction picture, the state $|\Psi(t)>_I$ satisfies the equation i $d|\Psi(t)>/dt=H_I(t)|\Psi(t)>_I$

a) Derive an equation for the interaction picture evolution operator $U(t; t_0)$ where $|\Psi(t)>_I = U(t; t_0)|\Psi(t_0)>_I$ with $U(t; t) = 1$.

b) Solve the equation you have derived in (a) for $U(t; t_0)$ when the Hamiltonian $H_I(t)$

satisfies $[H_I(t_1), H_I(t_2)] \neq 0$ for $t_1 \neq t_2$. Define all symbols you use, and show why your solution is true.

QMII-3

Consider a charge particle with mass m and charge q in a one dimensional simple harmonic oscillator potential $V(x) = \frac{1}{2}m\omega x^2$. Initial the particle is in the ground state. Between $0 < t < \pi/\omega$ the particle is subject to a constant electric field E perturbation. Using the first order time dependent perturbation theory, calculate the probability of the particle in the eigenstate $|n>$ at the end of a perturbation period $T = \pi/\omega$.

QMII-4

(a) Under the parity operation $\boldsymbol{\pi}$ how the coordinate $\boldsymbol{x}$, momentum $\boldsymbol{p}$, and angular momentum $\boldsymbol{L}$ transform?

(b)The ground state of a SHO $|0>$ is known to be parity even. Show that, in general, the excited state $|n>$ is parity even/odd, depending on whether n is an even or odd integer.

(c) Under time reversal operation $\boldsymbol{\theta}$, how the coordinate $\boldsymbol{x}$, momentum $\boldsymbol{p}$, and angular momentum $\boldsymbol{L}$ transform?

(d) A spin half particle is in a state $|\alpha >= a|+> +b|->$. What condition does one need to impose on the complex numbers a, b if the state is invariant under time reversal symmetry?

QMII-5

An isolated hydrogen atom has a hyperfine interaction between the proton and electron spins ($\mathbf{S}_1$ and $\mathbf{S}_2$, respectively) of the form $J\mathbf{S}_1.\mathbf{S}_2$. The two spins have magnetic moments $\alpha\, \mathbf{S_1}$ and $\beta\, \mathbf{S_2}$, and the system is in a uniform magnetic field $\boldsymbol{B}$. Consider only the orbital ground state.

(a) Find the exact energy eigenvalues of this system and sketch the hyperfine splitting spectrum as a function of magnetic field.

(b) Calculate the eigenstates associated with each level.

AstroI-1
Basic Astronomy

1. A new planet is discovered! Planet X is observed to orbit the sun every 300 years. What is the semi-major axis of Planet X's orbit in AU?

2. Planet X is in a circular orbit. Given that 1 AU = 1.5 × 1011 m, what is the distance to Planet X in meters when at closest approach to Earth?

3. When at closest approach to Earth, Planet X is observed to be 3.8 arcseconds in diameter. What is Planet X's diameter in meters?

4. What is the diameter of Planet X in Earth diameters and in AU? (The diameter of Earth is 1.3 × 107 m.)

5. Planet X is observed to have a small moon. This moon is observed to orbit the planet once per month at a distance of 15 Planet X diameters. What is the mass of Planet X in solar masses? (Assume that the mass of the moon is negligible in comparison.)

6. What is the mass of Planet X in kilograms and in Earth masses? (The mass of the sun is 2.0 × 1030 kg. The mass of Earth is 6.0 × 1024 kg.)

7. What is the average density of Planet X in kg/m3?

8. Based only on your calculated values for the diameter, mass, and density of Planet X, it is probably a:

A. Large comet
B. Large asteroid
C. Kuiper belt object similar to Pluto
D. Terrestrial planet similar to Earth
E. Jovian planet similar to Saturn
F. Jovian planet similar to Jupiter
G. Jovian planet similar to a giant Jupiter
H. Small star

AstroI-2
Consider a model star in which the density is a linear function of radius:
$\rho(r) = \rho_c\left[1 - r/R\right]$, where ρ_c is the central density and R is the total stellar radius at which P(R) = T(R) = 0.

a. Find an expression for the central density in terms of total radius R and total mass M

b. Use the equation of hydrostatic equilibrium and zero boundary conditions to find pressure as a function of radius. Write an expression for the central pressure in terms of R and M
c. What is the central temperature (assume ideal monoatomic gas equation of state).
d. Verify that this linear density model obeys the corollary to the virial theorem: $U = -\Omega/2$ where U is the total internal energy and Ω is the gravitational potential energy.

AstroI-3

Equilibrium of White Dwarfs

a. Derive the equation of hydrostatic equilibrium

b. Use dimensional arguments and the result of part a to derive the mass-radius relationship for a fully (non-relativistic) degenerate white dwarf

c. What is the slope of a white dwarf cooling track in the Log L, Teff plane (H-R diagram)? Make a plot of a white dwarf cooling track in this diagram. For this exercise you may assume a 0.6 solar mass white dwarf passes through solar temperature with a radius of 0.01 Rsun.

d. Now plot cooling tracks for white dwarfs of 1.1, 0.8 and 0.4 solar masses. Write the expression for the dependence of Log L on M for constant T.

AstroI-4

Virial Theorem

There is a commonly-cited corollary to the virial theorem as applied to spherical stars in hydrostatic equilibrium with ideal mono-atomic gas equations-of-state.

a. Re-derive this theorem under the following assumptions:

Ideal diatomic gas
Completely relativistic gas

b. The corollary to the Virial Theorem is often used to claim that a star powered by gravitational contraction will use ½ of the gravitational energy released to heat up, and the other half is radiated away. Explain why this is true for the ideal mono-atomi gas, and modify the statement for the two cases you have derived.

c. Physically speaking, why can there be no simple expression or statement like this for non-relativistic electron degenerate stars (i.e. white dwarfs).

AstroI-5
Lifetimes of Stars

a. Assuming the mass-luminosity relationship on the main sequence is $L \propto M^{3.5}$, derive a relationship for lifetime on the main sequence in terms of total M under the assumption that all stars have the same fraction of their H mass available for nuclear burning

b. Derive a similar relationship for stars powered by gravitational contraction alone (your expression will contain total mass and total radius).

c. Draw an H-R diagram that shows rough isochrones for co-eval populations 1, 5 and 10 Gyr after birth.

d. Now suppose there exist clusters made of pure iron stars with masses like those of normal stars. Describe the evolution of these stars as they follow Kelvin-Helmholtz contraction, and draw some isochrones for different age populations (please give the age in years for the isochrones you show, which obviously requires that you calculate the Kelvin-Helmholtz timescale with real numbers)

AstroII-1
Relaxation in a Galaxy Cluster

Consider an idealized galaxy cluster in which all galaxies have the same mass (10^{12} M_{sun}). The cluster contains 2000 galaxies within a radius of 2Mpc and has line-of-sight velocity dispersion 1000 km/s (somewhat like the Virgo Cluster).

(a) Draw a diagram of a two-galaxy encounter in the weak encounter limit (impulse approximation), in the rest frame of one of the galaxies. Call their relative velocity v. Show the impact parameter b, and use a long arrow to indicate the trajectory of the non-rest-frame galaxy.

(b) What is the change in velocity for each galaxy (make sure to answer for both)? Distinguish the parallel and perpendicular components. Note that $\int dx/(c_1+c_2x^2)^{3/2}$ $=2/(c1\sqrt{c2})$ for integration limits of $x=-\infty$ to $+\infty$.

(c) The weak encounter approximation breaks down if $\Delta v \sim v$, the typical 3D velocity of a galaxy in the cluster. What impact parameter does this breakdown occur at? Compare this “strong encounter” impact parameter to the typical distance between galaxies.

(d) Approximately calculate the two-body relaxation time for this cluster. Comparing this number to the age of the Universe, comment on how the cluster has achieved a relaxed, roughly spherical configuration.

AstroII-2
Closed Box Star Formation

(a) Demonstrate that an exponential star formation history results from the assumption that the star formation rate is proportional to the gas mass in a "closed box" system, i.e. that $\frac{dM_s}{dt}(t) = kM_g(t)$.

(b) Assume the metallicity Z increases as the gas is consumed with yield p, such that $Z(t) = -p\ln\frac{M_g(t)}{M_g(0)}$. Derive the time dependence of Z in terms of p and k. Compute the yield p for which the enrichment timescale for the metallicity to reach the solar value Z=0.02 is the same as the gas depletion timescale over which the gas drops by 1/*e*.

(c) If we allow for external gas infall (open box model), in which direction will the yield change from the value you computed in part b?

AstroII-3
A Young Massive Star Cluster

A massive ($7x10^7 M_{sun}$) star cluster with a light-weighted simple stellar population age of ~500 Myr is 6 kpc away on-sky from an elliptical galaxy with multiple tidal streams and shells, at the same redshift. The galaxy has dispersion ~135 km/s, equivalent to rotation velocity ~200 km/s.

(a) In the Chandrasekhar approximation, the drag force on this star cluster from dynamical friction with particles in a dark matter halo is $F_{df} = -\frac{4\pi G^2 M_{sat}^{\ 2}}{V_{sat}^{\ 2}} \rho \ln\Lambda$, where ρ is the mass density of dark matter particles and $\ln\Lambda \sim 3$. Prove that if the cluster starts on a nearly stable circular orbit, the time for the cluster to sink to the center of the galaxy is $t_{\sin k} = \frac{r^2 V}{18GM_{sat}}$, where V is the galaxy rotation velocity (which may be assumed constant at all radii).

(b) If the cluster stellar population is indeed "simple," is the light-weighted age surprising compared to the value of the sinking time, evaluated from the equation in part a? What if the stellar population turns out to be composite? Comment on how your answers are affected by the fact that the 6kpc distance is actually a lower limit due to projection on the sky.

(c) The cluster is virialized and its light profile resembles a cE like M32. Will it follow the Faber-Jackson relation of elliptical galaxies, or if not, how will it deviate? In your answer, explicitly discuss each assumption required to derived the Faber-Jackson relation.

AstroII-4
Spherical Gas Clouds

(a) A pressureless, uniform-density spherical gas cloud collapses under gravity in the free-fall timescale t_{ff}. Show that $t_{ff} = \sqrt{3\pi/(32G\rho)}$ using Kepler's 3rd Law $P^2 \propto a^3$, where ρ is the density of the cloud, P is the period of an orbit around a point mass, and a is the semi-major axis of an orbit around a point mass.

(b) Now suppose the cloud is not pressureless, but is supported by internal random motions with typical dispersion equal to the sound speed v. Write down an order-of-magnitude inequality describing the range of cloud sizes that remain unstable to collapse.

(c) A borderline stable molecular cloud has density ρ_A, size $l_A=2r_A$, and internal sound speed v_A. If cloud B is 6x smaller and 16x denser, prove that its sound speed v_B must be $0.67v_A$ to achieve the same borderline stability.

(d) If the speed of sound v_s is related to temperature T in a molecular gas cloud by $v=\sqrt{1.4k_BT/m}$ where m is the mass of a typical molecule and k_B is Boltzmann's constant, how do the wavelengths of peak blackbody emission for the two clouds in part c compare? What properties of the clouds suggest assuming blackbody emission is reasonable?

AstroII-5

Vertical Motion in a Spiral Galaxy Disk

A gas cloud plunges through a spiral disk that has scale height h_z = 350pc. The interaction creates young star clusters in the spiral disk, extending above it by ~100pc. The cloud emerges at z=500 pc.

Assume that at small heights z above the disk, the spiral disk potential takes the approximate form $\varphi = 4\pi G\rho_0 h_z^2 e^{-\frac{|z|}{h_z}} + 4\pi G\rho_0 h_z |z|$. For a realistic mass density ρ_0 ~ 0.1M_{sun}/pc^3 this means $\sqrt{\varphi(z=0)} = \sqrt{4\pi G\rho_0 h_z^2} \sim 25$ km/s.

(a) Expand the potential in a Taylor series around z=0 to show that the force equation is $F = -4\pi G\rho_0 z\left(1-\frac{z}{2h_z}\right)$.

(b) Will the newly formed star clusters experience simple harmonic motion? What is the third integral that is conserved for their orbits? What about the gas cloud?

(c) Use the Poisson equation appropriate for a thin-disk system to determine ρ.

(d) Show that the surface mass density is $2h_z\rho_0$.

Useful facts:

$F = Gm_1m_2/d^2$

G=4.3 x 10^{-6} kpc (km/s)^2/M_{sun}

1kpc = 1Gyr x 1km/s

Virial Theorem 2KE+PE=0

Poisson Equation $\nabla^2\varphi = -4\pi G\rho$

Wien's Law T = 3 mm·K / λ_{peak}

centripetal force F=mV^2/r (uniform circular motion)

Faber-Jackson Relation L $\propto \sigma^4$

Dynamical time $t_{dyn}=\sqrt{2}t_{ff}$

Crossing time t_{cross} = R/V = 1 Gyr (**R** in kpc/**V** in km/s)

Relaxation time t_{relax} = 0.1N / ln(N) t_{cross} = 10^6 yr x 0.1N/ln(N) x (**R** in pc/**V** in km/s)

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Quantum Mechanics, part I: May 10, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

QMI-1 Three Hermitian operators satisfy the following commutations relations: $[\hat{A}, \hat{C}] = [\hat{B}, \hat{C}] = 0$ and $[\hat{A}, \hat{B}] \neq 0$.
Show that the spectrum of the operator $\hat{C}$ contains degenerate eigenvalues.

QMI-2 Here is a guided proof that there are wave functions that oscillate back and forth in a one-dimensional harmonic oscillator without spreading:

(a) Consider a state of the form

$$|\psi\rangle = e^{-i\hat{p}\,\lambda/\hbar}\,|\phi\rangle\ ,$$

where $\hat{p}$ is the usual momentum operator, at time $t_0 = 0$. How is this state related to $|\phi\rangle$?

(b) Show that in the Schrödinger picture, the state vector $|\psi(t)\rangle$ for $t > 0$ is given by

$$|\psi(t)\rangle = e^{-i\hat{p}(-t)\,\lambda/\hbar}\,|\phi(t)\rangle\ ,$$

where $|\phi(t)\rangle$ is the Schrödinger-picture state that evolves from $|\phi\rangle$, and $\hat{p}(t)$ is the *Heisenberg-picture* operator that evolves from the usual momentum operator $\hat{p}$.

(c) Now let the system be a harmonic oscillator with oscillator frequency ω. Find $\hat{p}(-t)$ in terms of the $\hat{p}$ and the usual position operator $\hat{x}$. (*Hint:* For linear systems like the oscillator, the solutions to the Heisenberg equation of motion are the same as the classical solutions.)

(d) Now suppose $|\phi\rangle$ is the oscillator ground state. Use the fact that

$$e^{A+B} = e^A e^B e^{-1/2[A,B]} \qquad (1)$$

(if $[A, B]$ commutes with both A and B) and the form of the ground state wave function

$$\langle x|\phi\rangle \propto e^{-\frac{m\omega}{2\hbar}x^2} \qquad (2)$$

to show that while the wave function $\langle x|\psi(t)\rangle$ moves, it doesn't spread.

QMI-3 A particle of mass m is subject to the one-dimensional potential $U(x) = -\alpha\delta(x-a)$ for $x > 0$, and $U(x) = \infty$ for $x < 0$.

(a) Find the number of bound states as a function of the parameter $\alpha ma/\hbar^2$.

(b) Is the effective force acting between the particle and the wall repulsive or attractive?

QMI-4 Arbitrary spin operator. A non-interacting, spin-1/2 particle has an angular momentum component that is determined to be pointing in the $+\hat{z}$ direction. The spin is then measured along an arbitrary direction, $\hat{n}$.

(a) What is the total intrinsic angular momentum of this particle?

(b) Find the expectation value for the angular momentum measurement in the $\hat{n}$-direction.

(c) Assume for this and the subsequent question that the angle between the $+\hat{z}$ vector and $\hat{n}$ is $\pi/4$ radians. Find the probability that the angular momentum component measured in $\hat{n}$-direction is positive.

(d) A measurement of the angular momentum component pointing in the $\hat{n}$ direction is performed, and it is found to be greater than zero. What are the possible outcomes for another subsequent measurement of the angular momentum component along the z-axis and what are their probabilities?

Reminder:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

QMI-5 Non-rigid rotator. Consider a diatomic molecule consisting of identical atoms, each of mass m, and with a separation between nuclei of r_0 when the molecule is in the zero angular momentum state ($L = 0$). Assume that the binding force between the atoms can be modeled as a spring with spring constant k. Answer the following questions:

(a) The rotational excitation levels in most diatomic molecules have much lower energies than that of the vibrational excitation levels. Find a relationship between m, r_0, and k that has to be satisfied for this to be true.

(b) Assume that the molecule satisfies the condition that you found in part (a). It is placed in a gas with enough thermal energy to excite the rotational but not vibrational levels. By correcting for the fact that this molecule is *not* a rigid rotator, find the approximate frequency of a photon emitted during a transition from the first excited to ground rotational states.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Statistical Mechanics: May 10, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

SM-1 Consider the ideal Fermi gas where the single-particle eigenstates and eigenvalues of the Hamiltonian are s and ε_s, respectively.

(a) Based on the quantum properties of fermions, derive the grand partition function of such system with given fugacity z, volume V, and temperature T is given by

$$\mathcal{Z}(z,V,T)=\prod_s\left(1+ze^{-\varepsilon_s/k_BT}\right)$$

(b) Write down the relationship between $\mathcal{Z}$ and the grand canonical potential $\Phi = U - TS - \mu N = -PV$.

SM-2 Consider the internal rotational degree of freedom of a diatomic molecule with a moment of inertia I. Its Hamiltonian is given by $\hat{H}_{rot}=\hat{L}^2/(2I)$ and $\hat{L}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$ with $l=0,1,2,\ldots,\quad m=-l,-l+1,\ldots,l$.

(a) Write down the expression of its partition function associated with rotation (ignore the nuclear spin effect).

(b) Based on the result of (a), calculate $U_{rot}=U_{rot}(T)$ at the low temperature limit.

(c) Now consider molecular H_2 where the effect of nuclear spins has to be taken into account. Protons are spin-1/2 fermions and the two spins can form singlet and triplet states. Write down the partition function of rotation with the effect of the nuclear spin states included.

SM-3 Calculate the Joule-Thomson coefficient $(\partial U/\partial V)_T$ where U is the internal energy for a non-ideal gas described by the van der Waals' equation of state $P = RT/(V-B) - a/V^2$.

SM-4 The average energy of a system in thermodynamic equilibrium is $\langle E \rangle$.

(a) Show that the mean square of the energy deviation from its average value equals $\left\langle \left(E - \langle E \rangle\right)^2 \right\rangle = k_B T^2 C_V$.

(b) Estimate, for a system of N>>1 particles, the relative deviation $\left\langle \left(E - \langle E \rangle\right)^2 \right\rangle / \langle E^2 \rangle$ in the high-temperature limit.

SM-5 There have been recent attempts to interpret gravity as an entropic force (and possibly there are gaps in physical reasoning. But let us play along).

(a) Consider the entropic force $F\delta x = T\delta S$, where δx denotes an infinitesimal spatial separation, and fix δx by the Compton wavelength $\delta x = \hbar / (mc)$ for the particle of given mass m, for the infinitesimal increase of entropy $\delta S = 2\pi k_B$. Let us postulate the entropic change to be linear with the change in distance $\delta S = 2\pi k_B (mc/\hbar)\delta x$. Now, let us further adopt the famous formula $k_B T = \hbar a / (2\pi c)$ for the (Unruh) temperature associated with a uniformly accelerated (Rindler) observer with acceleration a. Derive the second law of Newton F=ma.

(b) Consider a point particle of mass M at the origin. One can associate an energy $E = Mc^2$ with this mass. But, let us assume that this energy is equal to the energy of N degrees of freedom at temperature T on the surface of the sphere of radius r, where $N = Ac^3 / (G_N \hbar)$ (with $A = 4\pi r^2$). Use the equipartition theorem, the Unruh temperature expression, and the result in (a) to derive Newton's law for the gravitational force: $F = G_N M m / r^2$.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Electromagnetism I: May 7, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

EMI-1 What is the potential in a rectangular region bounded by $0 < x < a$, and $0 < y < b$, given that the boundary conditions on the potential are that it vanishes on the two edges that have $y = constant$, that it is constant ϕ_0 (not zero) on the edge that has $x = a$, and that its derivative vanishes on the edge $x = 0$. Give a physical explanation for the mathematical content of your answer.

EMI-2 A thin spherical shell of radius R carries a uniform surface charge density σ. The shell is rotating along z axis with angular frequency ω.
(a) Write down the surface current density associated with the rotating charge, and the boundary condition for the magnetic field cross the shell.

(b) Show that the magnetic scalar potential is given by
$\Phi_m(r, \theta, \phi) = \frac{2}{3}\sigma\omega Rr cos\theta$ for $r < R$, and
$\Phi_m(r, \theta, \phi) = \frac{1}{3}\sigma\omega\frac{R^4}{r^2}cos\theta$ for $r > R$.

(c) Calculate the magnetic field both inside and outside R.

EMI-3 (a) Calculate the electric potential around a symmetric quadrupole comprising charge $-2q$ at the origin and charges $+q$ at distance a above and below on the z axis. Obtain the usual far field approximation for large r at the end.

(b) Now for same quadrupole inside a grounded conducting sphere of radius $b > a$, calculate the potential for all points $r > a$. Again obtain the far field approximation for all $r >> a$. (Hint: consider image charges.)

EMI-4 (a) Write the differential form of Maxwell's equations in vacuum with sources.

(b) When $B_i = \epsilon_{ijk}\partial^j A^k$ and $E_i = -\partial_i\phi + \partial_0 A_i$, show which two of Maxwell's equations are automatically satisfied.

(c) Show that $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2\vec{A}$.

Hint: Use $(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk}\partial^j A^k$ and $\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$.

(d) Give the gauge transformation on the vector potential A^k that leaves invariant the magnetic field B_i. Show that the gauge transformation $\phi \longrightarrow \phi - \frac{\partial \Lambda}{\partial t}$ combined with a gauge transformation on A^k leaves invariant the electric field E_i.

e) Write the remaining two Maxwell's equations, that are not automatically satisfied, in terms of the potentials A_i and ϕ in the gauge $\vec{\nabla} \cdot \vec{A} = 0$, when $A_i(\vec{x})$ and $\phi(\vec{x})$ are independent of time.

EMI-5 A thin uniform metal disk with mass density ρ is balanced on top of a much larger diameter conducting sphere of in a uniform gravitational field. The radius of the sphere is R and it never moves. Charge is slowly added to the sphere. At what total charge Q on the sphere would the disc starts to lift off from the sphere?

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Classical Mechanics: May 7, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

CM-1 Consider a mechanical system for which the potential energy $V(\vec{r_1}, \vec{r_2}...)$ is a homogeneous function of the coordinates $\vec{r_i}$ of degree n. Let us scale all the coordinates by a factor of α and the time by a factor of β.

(a) Show that for $\beta = \alpha^{1-\frac{n}{2}}$, the equation of motion is NOT changed under these scaling operations.

(b) Show that for $\beta = \alpha^{1-\frac{n}{2}}$, the same set of equations of motion permits a series of geometrically similar paths with the times of motion between corresponding points being given by the ratio $\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{n}{2}}$, where $\frac{l'}{l}$ is the ratio of linear dimensions of the two paths. What else is required besides scaling of the potential and time in order to permit self-similiar motion?

(c) Show that for harmonic oscillators, the period of oscillations is independent of their amplitudes (by using part b).

(d) Show that in free fall under gravity, the time of fall goes as the square root of the initial altitude (by using part b).

CM-2 If the mass and spring constant in a harmonic oscillator have a particular time dependence, one can arrive at the time-dependent Hamiltonian

$$H = f(t)\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\right],$$

where ω is a constant and $f(t)$ is the derivative of some other well behaved function $g(t)$.

(a) Write the Hamilton Jacobi equation for Hamilton's principal function $S(q, \alpha, t)$, where α is the "new" momentum.

(b) Solve the equation to find $q(t)$ in terms of the usual constants α and β. How would you describe the physical meaning of the constant α?

CM-3 A particle of mass m moves in one dimension (along x) under a potential

$$V = a^2x^4 - 2b^2x^2,$$

where a and b are constant parameters.

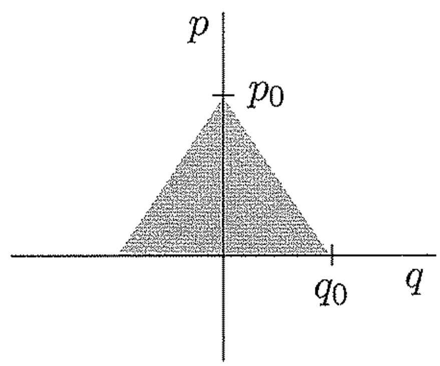
(a) Determine the locations of the equilibria.

(b) Find the frequency of small amplitude motion about stable equilibria.

(c) Find the exponential growth rate for small amplitude motion away from the unstable equilibrium.

(d) Derive the Hamiltonian and sketch the surfaces of constant energy in phase space.

CM-4 The Liouville theorem states that areas in phase space are conserved. Consider an ensemble of free particles and the initial $t = 0$ phase space distribution drawn below:



(a) Without using the Liouville theorem itself (unless you want to derive it), show that at time t the region has evolved into another region with the same area.

(b) Liouville's theorem can be proven by showing that the transformation from q_0, p_0 to $q(t), p(t)$ is canonical for any time t. Show explicitly that the transformation is canonical in this simple example.

CM-5 Consider a particle of mass m that is confined to the surface of a torus and is acted upon by a uniform gravitational acceleration g. Let the torus have minor radius b and major radius a. Positions on the torus are described by two angle coordinates, θ and ϕ. The angle ϕ is an azimuthal coordinate that circles the symmetry axis z (i.e., goes around the torus the long way) and θ goes through 2π as it circles the circular cross section of radius b. The transformation between cartesian coordinates and toroidal coordinates is

$$\begin{aligned} x &= (a + b\sin\theta)\cos\phi \\ y &= (a + b\sin\theta)\sin\phi \\ z &= b\cos\theta, \end{aligned}$$

and the line element (or metric) for the surface geometry of the torus is

$$ds^2 = b^2 d\theta^2 + (a + b\sin\theta)^2 d\phi^2,$$

which can be used to find the velocity (tangent vector) of any trajectory on the toroidal surface.

(a) Obtain the Lagrangian for motion on the toroidal surface.

(b) Determine the symmetries of the Lagrangian and the conserved quantities.

(c) Assuming there is some motion in the ϕ direction, obtain the effective potential for motion in θ.

(d) Assuming you are told that, under steady motion in ϕ at a certain rate Ω, the particle maintains a constant equilibrium angle θ_c. Given some θ_c, use the equations for equilibium to determine expressions for the values of the conserved quantities.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Quantum Mechanics, part II: May 10, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

QMII-1 Atomic Resonance. Consider a single electron that experiences a static central potential $V(r)$. We add a weak, external, time-varying magnetic field with a corresponding vector potential $\mathbf{A}(r,t)$. Use a specific case of the Lorentz gauge for this problem:

$$\nabla \cdot \mathbf{A} = \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$

(a) Write down the Hamiltonian for this system.

(b) Assume that the perturbation due to $\mathbf{A}$ is much smaller than the energy scale imposed by V. Use this property and the Lorentz Gauge condition to write the Hamiltonian as a sum of a static, unperturbed part and a smaller, time-dependent part. You may find the following vector identity useful:

$$\nabla \cdot (\mathbf{A}f) = (\nabla \cdot \mathbf{A})f + \mathbf{A} \cdot (\nabla f)$$

(c) Assume that the weak, time-varying potential is of the form:

$$\mathbf{A}(t) = \mathbf{A_0} \cos \omega t$$

where $\mathbf{A_0}$ may be assumed constant on the scale of the electron's wavefunction. Let a solution to the unperturbed potential from part (b) be written as:

$$\psi(t) = \sum_k c_k(t)\psi_k(t)$$

where ψ_k are the eigenfunction of the unperturbed Hamiltonian. Assume that the system is in eigenstate m of the unperturbed Hamiltonian at $t = 0$. Find the probability that the system will be in eigenstate n at a later time, given that $m \neq n$. You may collect all the time-independent coefficients (constants, expectation values, etc.) into one constant, N, that you are not required to evaluate.

(d) Assume that the value of the driving frequency, ω, is very close to the value of

$$\omega_{mn} = (E_m - E_n)/\hbar$$

and simplify the expression you found in part (c) further.

(e) Under the assumption from part (d), sketch the transition probability for the transition from state m to state n as a function of ω_{mn}. Comment on and discuss your result.

QMII-2 Morse Potential. A phenomenological formula that describes the interaction potential between two atoms in a diatomic molecule is the so-called *Morse Potential*:

$$V(r) = D(1 - e^{-\alpha(r-r_0)})^2$$

where r is the separation between the atomic nuclei.

(a) Sketch this potential and provide a physical interpretation of the parameters D and r_0.

(b) Provide a *qualitative* sketch of the energy-levels of this potential.

(c) Find a potential to approximate the given Morse potential and find the first non-zero perturbation theory correction to the ground state for that potential.

You may find the following information useful:

$$\int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n}, \quad n!! = n \times (n-2) \times (n-4) \times ...$$

$$\int_0^\infty x^{2n+1} e^{-px^2} dx = \frac{n!}{2p^{n+1}}$$

$$\psi_n(x) = (\frac{m\omega}{\pi\hbar})^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\nu) e^{-\nu^2/2} \; ; \nu = \sqrt{\frac{m\omega}{\hbar}} x$$

QMII-3 A particle of mass m is temporarily captured in a state with angular momentum $l > 0$ and energy $E > 0$ inside a sphericallly-symmetrical well of depth V and radius R. Neglecting the centrifugal barrier within the well, evaluate the half-life time τ of such a metastable state.

QMII-4 Particles in a well. Three identical, spin-$\frac{1}{2}$ particles, each with mass m, are trapped in an isotropic three dimensional harmonic oscillator well with a classical angular oscillation frequency ω. The only interaction that the particles experience amongst themselves is the coupling between their intrinsic magnetic dipole moments ($\vec{\mu}$). The potential energy of this coupling is equal to the dot product of the two dipole moments, multiplied by a constant, β. It does not depend on the distance between the particles.

(a) Write down the hamiltonian for this system.

(b) Assume that the spatial component of the system's wavefunction corresponds to its lowest energy state allowed by symmetry principles. Find the energies and degeneracies of all the allowed states in this case.

QMII-5 Consider scattering of a plane wave $|k>$ off a potential with a characteristic length a. It is know that the phase shifts δ_l for all spherical partial waves are given by the expression

$$\sin\delta_l = \sqrt{\frac{(ka)^l}{(2l+1)l!}}$$

(a) Considering the p partial wave only, what is the ratio of differential scattering cross section in the forward direction to that of the backward direction?

(b) If the first resonance scattering is observed for the p-wave at certain energy, what would be the s-wave scattering cross section at this particular energy?

(c) What would be the total differential cross section if all partial waves are included? Calculate the total scattering cross section for $ka = 1$.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Electromagnetic Theory II: May 10, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

EMII-1. A particle of mass m and Lorentz factor γ scatters off a particle of equal mass that was initially stationary. The collision is elastic and the first particle scatters off at an angle θ relative to its initial direction of motion. Determine the Lorentz factor γ' of this particle as a function of $\cos\theta$ and γ.

EMII-2. An undulator is a device for producing coherent electromagnetic radiation from a beam of relativistic electrons (i.e., free electron laser). In an undulator, a relativistic electron passes through a region with alternating magnetic field direction. The alternating magnetic field causes the electron to wiggle in the transverse direction, and thereby radiate.

Assume the magnetic field in the device varies sinusoidally in the y-direction,

$$\vec{B} = (0, B_0 \cos(kz), 0),$$

with the magnet spacing related to k. Let the electron's velocity be primarily along the z-direction but perturbed by the magnetic field,

$$\vec{v} = (u(t), 0, v).$$

Here the (longitudinal) z-component is $v \simeq c$ and is unaffected by the magnetic field at first order.

(a) In whatever frame you choose, use the equations of motion to express the time dependence of the x and t components of the four velocity.

(b) What is the solution for the time dependence of the Lorentz factor?

(c) In whatever frame you choose for the calculation, compute the total average radiated power from a single relativistic electron as seen in the lab frame of the undulator.

EMII-3. A general expression for the spectral and angular distribution of energy radiated from a relativistic electron is

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\vec{n}\times(\vec{n}\times\vec{\beta})\exp[i\omega(t'-\vec{n}\cdot\vec{x}(t')/c)]\right|^2,$$

which is derived using the Lienard-Wiechart expression for the radiative part of the electric field.

A neutral particle like the Z^0 can decay into an electron-positron pair. If the Z^0 is at rest when it decays, the pair of particles fly in opposite directions but with equal speeds β.

Let the electron have velocity

$$\vec{\beta}_{e^-} = (0,0,\beta) \qquad \text{for} \quad t' > 0,$$

and thus have position vector

$$\vec{x}_{e^-}(t') = (0,0,c\beta t') \qquad \text{for} \quad t' > 0.$$

The positron's velocity and position vector are similar but with the sign of β reversed. Let the observation direction be taken to be $\vec{n} = (\sin\theta, 0, \cos\theta)$.

(a) Calculate the appearance radiation for pair production.

(b) Consider the nonrelativistic limit of this result. What does the angular dependence and (lowest) power of velocity in this limiting expression suggest?

EMII-4. Recalling Faraday's and Ampere's laws for a medium with non-trivial permittivity,

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{1}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}, \tag{2}$$

assume the presence of a transverse plane electromagnetic wave $\vec{E} = \vec{E}_0\exp(i\vec{k}\cdot\vec{x} - i\omega t)$. Let the permittivity be scalar and given by

$$\epsilon = 1 - \frac{\omega_p^2}{\omega(\omega+i\nu)}. \tag{3}$$

(a) Derive the dispersion relation.

(b) Using the dispersion relation and assuming a real driving frequency ω with $\omega > \omega_p$ and $\omega \gg \nu$, calculate the damping distance (or e-folding distance) for a plane wave propagating in this medium.

EMII-5. An electron of charge e and mass m moves in a circular orbit under the Coulomb force produced by a proton. The average potential enenergy $< V(r) >$ is related to the total energy by $E =< V > /2$. Suppose, as it radiates, the electron continues to move on a circle.

(a) Show that the power radiated is given by $-\frac{dE}{dt} = \frac{2e^2}{3c^3}\left(\frac{e^2}{mr^2}\right)^2$.

(b) Show that it takes the electron $t = \frac{m^2c^3r_{in}^3}{4e^4}$ to hit the proton if it starts from an initial radius of r_{in}. Assume you have never heard of quantum mechanics.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

<u>Astrophysics I: May 10, 1:30pm-4:30pm</u>

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problems number and your PID, but not your name.

1. Determine the fraction of hydrogen atoms that are ionized at the center of the sun, assuming ionization equilibrium, $T = 15.8$ million K, and $n_e = 6.4 \times 10^{31}\ \mathrm{m}^{-3}$. Does your result agree with the fact that practically all of the sun's hydrogen is ionized at the sun's center? What are reasons for any discrepancy? Suppose the star has twice the metal content of the sun. Would the level of ionization in its center be higher or lower than in the center of the sun? Why?

2. Approximate a white dwarf of mass M and radius R as a degenerate core surrounded by an ideal gas atmosphere.

   (a) Show that the atmospheric pressure as function of temperature is in general

$$P(T) = \left(\frac{64\pi ac\mathcal{R}G}{51\kappa_0\mu}\right)^{1/2}(\Upsilon)^{1/2}T^{14/4}$$

   for mass/luminosity ratio Υ.

   (b) From this expression, show that the temperature profile throughout this atmosphere $r < R$ is

$$T(r) = \frac{4}{17}\frac{\mu}{\mathcal{R}}GM\left(\frac{1}{r} - \frac{1}{R}\right)$$

3. The figure below show a schematic of a (theorist's) H-R diagram left blank except for the sun.

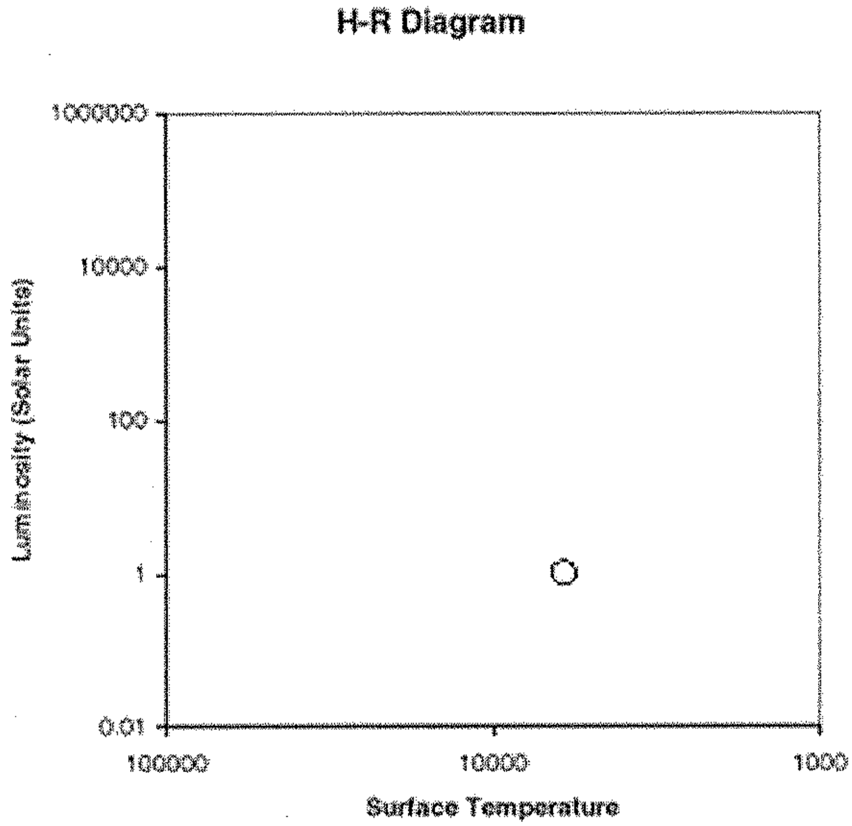


Figure 1: H-R diagram

(a) Sketch the main sequence in the diagram.

(b) Explain the procedure for comparing calculated luminosities and temperatures of stellar models to the observable quantities (color-magnitude).

(c) Assuming the mass-luminosity relationship on the main sequence is $L \propto M^{3.5}$, derive a relationship for lifetime on the main sequence under the assumption that all stars have the same fraction of their H available for nuclear burning. Then draw a + at the location of a star with ~1/700 the lifetime of the sun.

(d) Draw a * at the location of a star that has 100 times smaller luminosity than the sun and 10 times smaller radius.

(e) Draw an arrow showing the approximate direction the sun would move in the diagram if it cooled without changing its radius.

4. Consider a planetary transit across the disk of another star.

   (a) Using geometry and a relationship for limb darkening, plot the shape of the light curve.

   (b) What is the transit duration of a Jupiter analog orbiting at 0.3 AU across the center of the disk of a G2 main-sequence star? (Jupiter is $10\times$ the diameter of the Earth.)

   (c) What is the maximum eclipse depth in % visible wavelength light attenuation for b)?

   (d) Assuming that the planet has temperature 900 K, what is the ratio of IR flux at 3 micron wavelength of the planet to star in b)?

5. Estimate the duration of the helium burning phase of the sun (i.e. when it is on the horizontal branch). Assume:

   (a) Its luminosity on this branch will be 100 times higher than it is now.

   (b) Each reaction fusing 3 helium nuclei into 1 carbon produces 1.8×10^{-12} J, which is 40% the energy produced in the fusion of 4 hydrogen to 1 helium.

   (c) The sun will start helium burning with 10 % of a solar mass of helium, and will fuse essentially all of it. Remember that each helium nucleus is about four times as massive as a hydrogen nucleus (helium mass $= 6.7 \times 10^{-27}$ kg).

   (d) Using these assumptions, how long will the Sun burn helium? You may give your answer in years or in relation to the main sequence lifetime of the sun, but please indicate which you mean.

2010 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Astrophysics II: May 10, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problems number and your PID, but not your name.

ASTROII-1. Equation of State of a Degenerate, Ideal Fermi Gas

Consider a completely degenerate, ideal electron gas.

a. Write down an expression for the electron number density n_e in terms of the distribution function in phase space. Solve it, yielding n_e as a function of $x = p_F/m_e c$.

b. Assume that the mass density ρ is dominated by non-degenerate ions. Write down an expression for ρ as a function of x.

c. Write down an expression for the electron pressure P_e in terms of the distribution function. Solve it, yielding P_e as a function of x.

d. Series expand P_e in the relativistic limit. Keep only the leading term.

e. What then is the equation of state in the relativistic limit?

$\int_0^x \frac{x^4 dx}{(1+x)^{1/2}} = \frac{3}{8}\{x(1+x^2)^{1/2}(2x^2/3-1) + \ln[x+(1+x^2)^{1/2}]\}$

$\int_0^x (1+x)^{1/2} x^2 dx = \frac{1}{8}\{x(1+x^2)^{1/2}(1+2x^2) - \ln[x+(1+x^2)^{1/2}]\}$

ASTROII-2. Chandrasekhar Limit

Consider a white dwarf of radius R consisting of N fermions.

a. Write down an approximate expression for the typical distance q between fermions as a function of N and R. Ignore factors of order unity.

b. Using this and the uncertainty principle, write down an approximate expression for the typical momentum p of a fermion as a function of N and R. Ignore factors of order unity.

c. Using this, write down an approximate expression for the typical kinetic energy E_{KE} of a fermion in the relativistic limit as a function of N and R.

d. Write down an approximate expression for the typical gravitational potential energy E_{PE} of a fermion as a function of N and R. Keep in mind that although the pressure is dominated by electrons, the mass is dominated by baryons. Ignore factors of order unity.

e. Write down an approximate expression for the typical total energy E of a fermion as a function of N and R.

f. If N is small, $E > 0$ and can be minimized by increasing R until the fermions become non-relativistic. If N is large, $E < 0$ and can be minimized by decreasing R (i.e., the white dwarf collapses). Consequently, determine an approximate expression for the largest value of N that a white dwarf can have without collapsing. Evaluate it.

$\hbar = 1.1 \times 10^{-27}$ erg s

$c = 3.0 \times 10^{10}$ cm s^{-1}

$G = 6.7 \times 10^{-8}$ cm^3 g^{-1} s^{-2}

$m_B = 1.7 \times 10^{-24}$ g

ASTROII-3. White Dwarf Cooling

Consider a carbon white dwarf of mass M and of interior temperature T that is in excess of the crystallization temperature.

a. Write down an expression for the thermal energy per ion as a function of T.

b. Write down an expression for the total thermal energy of the white dwarf as a function of T and M.

c. Using this, write down an expression for the luminosity L of the white dwarf as a function of T and M.

d. Photon diffusion from the interior to the surface implies that:

$$L = (2 \times 10^6 \, \mathrm{erg/s}) \left(\frac{M}{M_\odot} \right) T^{7/2}. \tag{1}$$

Using this, write down a differential equation for T as a function of time.

e. Solve it assuming that the initial temperature is much greater than T. Write down an expression for the age τ of the white dwarf as a function of T.

f. What is the interior temperature of a 0.65-$M_\odot$ white dwarf of luminosity 10^{31} erg? What is its age in years?

$k = 1.4 \times 10^{-16}$ erg K^{-1}

$m_u = 1.7 \times 10^{-24}$ g

$M_\odot = 2.0 \times 10^{33}$ g

ASTROII-4. Photodissociation

Consider the photodissociation of ^{56}Fe before a Type II supernova:

a. Each ^{56}Fe nucleus dissociates into 13 alpha particles and 4 neutrons. Write down an expression relating their chemical potentials.

b. For a Maxwell-Boltzmann gas:

$$n_i = g_i \left(\frac{m_i kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{\mu_i - m_i c^2}{kT}\right). \qquad (2)$$

where $g_{Fe} \approx 1.4$, $g_\alpha = 1$, and $g_n = 2$. Write down the Saha equation, where:

$$Q = (13m_\alpha + 4m_n - m_{Fe})c^2 = 124.4\,\text{MeV}. \qquad (3)$$

c. Assuming that ^{56}Fe is the most abundant heavy nucleus, write down an expression relating n_α and n_n.

d. Using this, write down an expression relating the mass density ρ and the temperature T when half of the mass has been dissociated.

e. What is the mass density at which this occurs if $kT = 1$ MeV ($= 1.6 \times 10^{-6}$ erg)?

$m_u = 1.7 \times 10^{-24}$ g

$\hbar = 1.1 \times 10^{-27}$ erg s

ASTROII-5. Relativistic Beaming

Consider the jet of a very low-redshift ($z \ll 1$) gamma-ray burst. Assume that its bulk Lorentz factor as a function of observer-frame time is given by:

$$\Gamma = 100 \left(\frac{t}{1\,\mathrm{min}} \right)^{-3/8} \tag{4}$$

a. Assume that the jet is 0.2 radians across, that it is not expanding laterally with time, and that its center is pointed directly at us. At what observer-frame time, in hours, will the jet begin to fade in brightness more quickly?

b. At what observer-frame time would this occur if the same gamma-ray burst were at redshift 6.3?

c. Again assume that $z \ll 1$, but that the jet is not pointed directly at us. Assume that its center is pointed 0.2 radians away from us. At what observer-frame time, in hours, will the jet begin to brighten?

d. Since this is much longer than the jet's gamma-ray emitting phase, such events are likely missed by gamma-ray spacecraft, but might be picked up as "orphan" afterglows in optical surveys.

Suppose that in a 1-minute exposure you could detect regular afterglows typically for 1 hour, but orphan afterglows typically only for 15 minutes, since they take a while to brighten. Also assume that to this detection limit, one orphan afterglow appears (somewhere) in the sky every day. If your field of view is a large 1 square degree and you take 1-minute exposures for 10 hours each night, how many nights will it take you to detect an orphan afterglow?

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Classical Mechanics: May 8, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

CM-1 Nambu mechanics, a generalized Hamiltonian mechanics, introduces a triplet of dynamical variables $\vec{r} = (r_1, r_2, r_3)$, and two Hamiltonian-like functions H and G of $\vec{r}$. The equations of motion are given by

$$\frac{dr_i}{dt} = \epsilon_{ijk}(\partial_j H)(\partial_k G), \tag{1}$$

where we have used Einstein's summation convention, ϵ_{ijk} is the three-dimensional Levi-Civita symbol, and $\partial_j \equiv \frac{\partial}{\partial r_j}$, or in vector notation (with $\dot{a} \equiv \frac{da}{dt}$),

$$\dot{\vec{r}} = (\nabla H) \times (\nabla G). \tag{2}$$

(a) Show that, for any function $F = F(\vec{r})$,

$$\dot{F} = \epsilon_{ijk}(\partial_i F)(\partial_j H)(\partial_k G). \tag{3}$$

(b) Show that $\dot{\vec{r}}$ is divergenceless, i.e.,

$$\nabla \cdot \dot{\vec{r}} = 0. \tag{4}$$

(Side remark: This property is necessary to lead to Liouville's theorem.)

(c) An example of Nambu mechanics is provided by taking the dynamical variables to be $\vec{L}$, the components of angular momentum of a rigid body in the body-fixed frame

$$\vec{L} = (L_1, L_2, L_3), \tag{5}$$

and taking the functions H and G to be

$$H = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2), \tag{6}$$

$$G = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right), \tag{7}$$

where $I_{1,2,3}$ are the principal moments of inertia. Show that the equations of motion

$$\dot{\vec{L}} = (\nabla H) \times (\nabla G), \tag{8}$$

are Euler's equations for a free rigid top.

CM-2 Let the following equations represent a possible canonical transformation

$$Q = \alpha(p^2 + q^2)\cos\beta - \tan^{-1}\left(\frac{p}{q}\right)\sin\beta, \tag{1}$$

$$P = \alpha(p^2 + q^2)\sin\beta + \tan^{-1}\left(\frac{p}{q}\right)\cos\beta, \tag{2}$$

where α and β are constants.

Determine whether there exist values of α and β that make this transformation canonical.

CM-3 An infinite linear series of point masses, each with mass $m_j = m$, are connected by springs, each with spring constant k. The equilibrium length of each spring, and hence separation of each pair of masses, is b. So, in equilibrium the positions of the masses are $x_j0 = jb$.

(a) Obtain the potential and kinetic energy expressions, and from them the Lagrangian. Using $\eta_j = x_j - jb$ as coordinates to describe small oscillations, derive the equation of motion of each mass.

(b) Assuming harmonic time dependence $\eta_j = a_j e^{-i\omega t}$, where a_j is the amplitude of the jth mass, reduce the equations of motion to a recursion relation.

(c) Assuming the amplitudes also have harmonic dependence in the discrete space coordinate x_j, solve the system of equations for travelling waves and find the dispersion relation connecting frequency ω and wavenumber κ.

CM-4 A hemispherical bowl with radius R has a narrow groove cut in it from the rim down through the bottom of the bowl and back up to the rim on the opposite side. A small piece of ice with mass m slides without friction in the groove. The bowl is placed at the center of a turntable, which rotates with angular velocity ω.

(a) Define a generalized coordinate that specifies the location of the ice, and find a conserved quantity for the system.

(b) When the turntable is rotated slowly the ice oscillates around the bowl's bottom. Find the angular frequency Ω of small-amplitude oscillations.

(c) The turntable is sped up until it reaches a critical speed ω_c at which the oscillation around the bottom of the bowl becomes unstable. Find ω_c.

(d) For $\omega > \omega_c$, two other points in the groove become centers of stable oscillations. What are these two points.

CM-5 In the lab frame, a particle of mass m_1 moving with velocity $\vec{v}_1$ collides elastically with a particle of mass m_2 that is at rest. Assume that the dynamics is non-relativistic. Take without loss of generality that the scattering is confined to the x-y plane and that the initial motion is in the x direction.

(a) Work out the relationship between θ', the angle particle 1 is scattered into relative to the initial direction as seen in the center of momentum frame, and θ, the corresponding angle seen in the lab frame.

(b) It is observed in the lab frame that scattering never occurs at an angle beyond θ_{max}. Find a relationship between θ_{max} and the mass ratio m_2/m_1. What is $\cos\theta'$ when $\theta = \theta_{max}$?

(c) Assume that the differential cross section in the center of momentum frame, $d\sigma/d\Omega'$, is known. Relate this to the differential cross section measured in the lab frame, $d\sigma/d\Omega$. How does the latter behave near θ_{max}?

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Electromagnetism I: May 8, 1:30pm-4:30pm

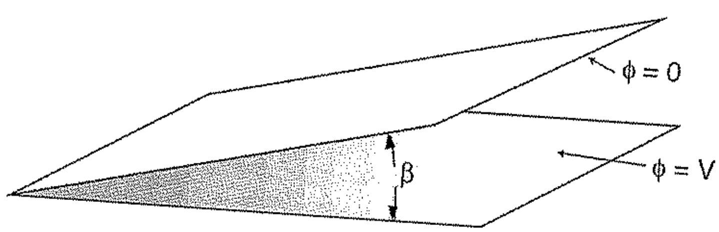
Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

EMI-1 An infinite slab of thickness ℓ is made of metal with conductivity σ and $\mu = \epsilon = 1$. Everywhere outside of it is a time-dependent magnetic flux density that is uniform in space, pointing parallel to the slab: $\mathrm{B} = \mathrm{B}_0 \cos(\omega t)$. Assuming that the fields are quasi-static, find the magnetic flux density everywhere inside of the slab.

EMI-2 A dielectric sphere of radius a with polarizability α is located at a distance $r \gg a$ from a conducting sphere of radius b which is maintained at a potential V. Find the force acting on the dielectric sphere.

EMI-3 A plane is held at a potential $\phi = \mathrm{V}$ while another plane, inclined at an angle β is held at $\phi = 0$ (as shown below). What is the potential between the planes, but near where they join? Ignore the edges.



EMI-4 A semispherical metallic bulge of radius a is placed on a infinite conducting plan held at the ground potential. A point charge q is placed on the position $(r = 2a, \theta = \pi/4, \phi = 0)$ relative to the center of the sphere. Calculate the electrostatic interaction energy between the charge and the conductor.

EMI-5 An infinite long cylinder of radius a and magnetic permeability μ, is placed in a uniform external magnet field H_0 which is perpendicular to the cylindrical axis. What is the magnetic field H at the center of the cylinder.

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Statistical Mechanics : May 8, 1:30pm-4:30pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

SM-1 The goal of this problem is to compare the low-temperature behavior of an ideal fermion gas to that of a classical ideal gas. You may treat either gas as non-relativistic.
(a) Under what condition in terms of the particle mass m, the number of particles N and the volume V can a gas be considered a classical ideal gas? Explain!
(b) Explain the meaning of the Fermi energy EF and temperature TF, why is there no corresponding concept for classical gases?
(c) Compare, quantitatively, the difference between an ideal Fermi gas and an ideal classical gas, for temperatures much smaller than TF, in terms of: (1) The kinetic energy per particle; (2) The heat capacity per particle and (3) Pressure and (4) Temperature dependence of the pressure.

SM-2 The temperature of a long vertical column of a particular substance is T everywhere. Below a certain height $h(T)$ the substance is solid, whereas above $h(T)$ it is in a liquid phase. Calculate the density difference $\Delta\rho = \rho_s - \rho_l$ between the solid and liquid ($|\Delta\rho| \ll \rho_s$), in terms of L (the latent heat of fusion per unit mass), ρ_l, dh/dT, T, and g, the acceleration due to the gravity. (Hint: use the Clausius-Clapeyron relation for phase separation line.)

SM-3 Find the internal energy and derive the equation of state of the gas of N massless and spinless relativistic particles with the dispersion $E = cp$ confined to a three-dimensional volume V.

SM-4 Find a specific heat of the system of N non-interacting magnetic ions of spin $S = 1$ which, due to an interaction with the surrounding non-magnetic ions, have their $m = \pm 1$ states degenerate with the energy $E = \epsilon$, while the state $m = 0$ has energy $E = 0$.

SM-5 (For this problem, you can set $c = 1, \hbar = 1, k_B = 1$, and ignore multiplicative factors of order 1.) Consider a perfect gas of N massless particles obeying Boltzmann statistics in volume $V \sim R_H^3$ at temperature $T \sim R_H^{-1}$.

(a) Find the thermal wavelength λ.

(b) Find successively the partition function, the free energy, and the entropy S. Show that $S \simeq N\left(ln(\frac{V}{N\lambda^3}) + O(1)\right)$, where $O(1)$ means of order 1.

(c) For $N \gg 1$, show that S is nonsensically negative.

(d) Now imagine that the particles are DISTINGUISHABLE. What then is the partition function? Show that, (even) for $N \gg 1$, S is non-negative (or, at least, not obviously negative).

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Quantum Mechanics I: May 8, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

QM1-1 Consider an isotropic harmonic oscillator in two dimensions (in units with $\hbar = m = 1$ to make things easier):

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{\omega^2}{2}\left(x^2 + y^2\right) = \omega\left(a_x^\dagger a_x + a_y^\dagger a_y + \frac{1}{2}\right)$$

with

$$a_x = \frac{1}{\sqrt{2}}(\sqrt{\omega}x + i\frac{p_x}{\sqrt{\omega}}), \quad a_x^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega}x - i\frac{p_x}{\sqrt{\omega}}), \quad \text{etc.}$$

(a) What are the energies of the eigenstates? What is the degeneracy of the ground state, the first excited state, the n^{th} excited state?

(b) Define

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_x \mp i a_y\right), \quad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_x \pm i a_y\right).$$

Show that

$$H = \omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + \frac{1}{2}\right), \quad L = \left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$$

Here $L \equiv L_z$ is the angular momentum in two dimensions.

(c) Show that the eigenstates of H can be chosen to be eigenstates of L as well. What values of L are associated with the ground state, the lowest excited state(s), the n^{th} excited state(s)?

QM1-2 (a) Show that if H is the Hamiltonian of a system, A is any operator, and $|\Psi\rangle$ is a state of definite energy, then

$$\langle\Psi|[H, A]|\Psi\rangle = 0.$$

(b) Consider a particle moving in a one-dimensional potential $V(x) = kx^n$, with k real and n an integer. By taking A above to be xp, derive the quantum virial theorem, a relation between $\langle\Psi|T|\Psi\rangle$ (the average kinetic energy) and $\langle\Psi|V|\Psi\rangle$.

(c) Show that for $V = 1/2m\omega^2 x^2$

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{E}{\omega}$$

QM1-3 Consider a particle that has three energy eigenstates $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$. Consider also a state permutation operator R that is defined by the following three operations:

$$R|\psi_1\rangle = |\psi_2\rangle; R|\psi_2\rangle = |\psi_3\rangle; R|\psi_3\rangle = |\psi_1\rangle$$

(a) Find the matrix representation of R and its eigenvalues and normalized eigenvectors. Use the basis defined by the energy eigenstates.

(b) Is R an observable? Why or why not?

(c) Suppose you know nothing about some operator except that it is both unitary and Hermitian. What eigenvalues can it possibly have?

QM1-4 (a) Show that an for arbitrary vector operator $\vec{F}$

$$\langle\alpha, jm|F_q|\alpha, jm'\rangle = f(j)\,\langle\alpha, jm|J_q|\alpha, jm'\rangle\,,$$

where q is an arbitrary component and $f(j)$ depends on j (and other quantum numbers α), but *not* on m or m'.

(b) Show that

$$\langle\alpha, jm|\vec{F}\cdot\vec{J}|\alpha, jm\rangle = f(j)\,j(j+1)\hbar^2\,,$$

where $f(j)$ is the same function as above. Use this result to prove a special case of the "Landé projection theorem":

$$\langle\alpha, jm|F_q|\alpha, jm'\rangle = \frac{\langle\alpha, jm|\vec{F}\cdot\vec{J}|\alpha, jm\rangle}{\hbar^2 j(j+1)}\langle\alpha, jm|J_q|\alpha, jm'\rangle\,.$$

QM1-5 Consider a "periodic" one-dimensional potential, satisfying $\langle x|\,V\,|x'\rangle = \langle x+a|\,V\,|x'+a\rangle$ for some period a. Let $\mathcal{T}(a) \equiv e^{-ipa/\hbar}$, with p the momentum operator. $\mathcal{T}(a)$ is a "displacement" operator.

(a) Show that

$$\mathcal{T}(a)V\mathcal{T}^{-1}(a) = V\,.$$

and that, as a consequence, one can find a complete set of eigenstates of H that are also eigenstates of $\mathcal{T}(a)$.

(b) Show that for any of these energy eigenstates $|\Psi\rangle$,

$$\langle x - a|\Psi\rangle = \lambda \langle x|\Psi\rangle$$

for some λ.

(c) Show that for the wave function $\psi(x)$ to be finite at $x = \pm\infty$, λ must be given by $\lambda = e^{iKa}$, for some real K. Congratulations, you've just proved Bloch's theorem for periodic potentials:

$$\psi(x - a) = e^{iKa}\psi(x)\,.$$

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Electromagnetism II: May 11, 9:00am-12:00pm

Choose 3 out of 5 problems for each subject

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

EM2-1 Consider a particle of electrical charge e and mass m, in the presence of a stationary magnetic charge g at the origin of the coordinate system. Let $\vec{r}$ be the coordinate of the electrically charged particle and $\vec{v}$ be its velocity.

(a) Find the equation of motion for the electrically charged particle.

(b) Find the associated moment equation, i.e., $\vec{r} \times m\frac{d\vec{v}}{dt}$.

(c) Identify the resulting constant of motion. (You may want to recall $\frac{dr}{dt} = \frac{\vec{r}\cdot\vec{v}}{r}$.)

(d) Assuming that all components of angular momentum are quantized, derive Dirac's quantization of charge, $\frac{eg}{c} = n\hbar$, where n is an integer or half-integer. ($\hbar$ is Planck's constant and c is speed of light.)

EM2-2 (For this problem, you may set speed of light $c = 1$ and ignore multiplicative constants of order 1.)

(a) Any accelerating charged particle radiates. The total radiated power P is given by the Larmor formula. Write down the Larmor formula for a non-relativistic particle of charge e.

(b) Apply (a) to a particle of charge e and mass m moving in a Hooke's law potential (a linear oscillator) with natural frequency ω_o. Find P.

(c) Recall that for such a motion, the time-averaged kinetic (T) and potential energy (V) satisfy, $\bar{T} = \bar{V} = E/2$, with E being the total energy.

Show that the power radiated, averaged over one cycle, is $\bar{P} = \gamma E$. Find γ.

(d) Show that the mean lifetime τ of the motion is given by $1/\gamma$. Find τ.

EM2-3 Consider the synchrotron radiation produced by a very high energy particle of charge e moving in a circle with frequency ω_o.

(a) Show that the radiation is concentrated near the forward direction in a narrow angle range $\phi \sim \sqrt{1-\beta^2}$ (where $\beta = v/c$) by recalling that the vector potential (in the Lorenz gauge) produced by a moving charged particle with velocity $\vec{v}$ is given by $\vec{A}(\vec{r},t) = \frac{e\vec{v}(t')/c}{|\vec{r}-\vec{r}(t')|(1-\vec{n}\cdot\vec{v}(t')/c)}$ where $\vec{r}(t')$ is the position vector of the particle at the emission time t' while t is the detection time; $\vec{n}$ is the direction of observation.

(b) Using (a), show that a typical frequency emitted is $\omega_e \sim \frac{\omega_o}{\sqrt{1-\beta^2}}$.

(c) Show that the detection time intervals and emission time intervals are related by $dt \sim (1-\beta^2)dt'$.

(d) Using (b) and (c), show that the detection frequency is given by $\omega_d \sim \frac{\omega_o}{(1-\beta^2)^{3/2}}$, which is much larger than ω_o.

EM2-4 Consider an isolated polarizable "molecule" located at $\vec{x} = 0$. Assume that an electron (charge $-e$) can be displaced from the molecular center by an electric field. Let the mass of the charge be m and assume the equation of motion is

$$\frac{d^2\vec{x}}{dt^2} + \omega_0^2\vec{x} = \tau\frac{d^3\vec{x}}{dt^3} - \frac{e}{m}\vec{E}(t),$$

i.e., a damped, driven harmonic oscillator with driving force given by an external electric field $\vec{E}(t)$ and a "self-force," radiative damping term.

Let the electric field be that of a plane wave of frequency ω propagating in the z direction with field oriented in the x direction with amplitude $E(t) = E_0 \exp(-i\omega t)$.

The classical electron radius is defined as $r_0 = e^2/(mc^2)$ and the Thomson cross section is $\sigma_T = (8\pi/3){r_0}^2$. The radiative damping time is $\tau = 2r_0/3c$.

(a) Ignore homogeneous solutions to the equation of motion and calculate the driven response.

(b) Use Larmor's formula to compute the total power scattered from the molecule and derive from this the total cross section as a function of frequency ω.

(c) The damping is important near resonance. Give expressions for the cross section at resonance and the width $\Delta\omega$ of the resonance.

EM2-5 Consider a simple classical, one-oscillator model for the interaction between an electromagnetic wave and an atomic gas. In this model the permittivity is

$$\epsilon = \frac{k^2c^2}{\omega^2} = n^2 = 1 - \frac{\omega_p^2}{\omega^2 + i\nu\omega - \omega_0^2}, \tag{1}$$

where ω_0 is the natural frequency of the atom (i.e., the frequency of the atom's spectral lines), ν is a damping rate associated with spontaneous emission, k is the wave number, n is the index of refraction, and ω_p is the plasma frequency (proportional to the square root of the number density of atoms).

(a) Obtain expressions for the real and imaginary parts of the permittivity.

(b) Resonance occurs at a frequency near $\omega \simeq \omega_0$. Obtain an estimate of the size of the imaginary part of ϵ at resonance (which is related to the absorption of the wave).

(c) Show that the resonance lies between $\omega_1^2 = \omega_0^2 - \omega_0\nu$ and $\omega_2^2 = \omega_0^2 + \omega_0\nu$.

(d) Show the existence of anomalous dispersion in the real part of ϵ in the region $\omega_1 \leq \omega \leq \omega_2$. What is the behavior of ϵ for $\omega \ll \omega_0$ and for $\omega \gg \omega_0$?

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Quantum Mechanics II: May 11, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

QM2-1 Consider a 'quantum pendulum' represented by a particle of mass m at the end of a massless rod of length R. The other end of the rod is fixed and the rod is constrained to move frictionlessly on a plane that is perpendicular to the surface of the earth, such that the particle experiences a uniform gravitational acceleration, g. In the large radius regime where the quantum fluctuations of the rod about its equilibrium vertical position are small, find the first quantum correction measured with respect to the classical ground state energy ($E_0 = -mgR$).

QM2-2 Linear Stark Effect: Consider the effect of a uniform electric field E (along z-direction) on the atomic spectrum of the H atom, without considering the electron spin.

(a) For $n = 2$ states, write down the perturbation matrix due to the electric field, specify all zero-matrix elements and explain why they are zero;

(b) Calculate the energy spectrum by diagonalizing the perturbation matrix;

(c) Schematically draw the energy levels, and indicate the associate eigenstates.

QM2-3 At $t = 0$ a one-dimensional oscillator described by a *time−dependent* potential $V(x,t) = m\omega^2(x - a\cos\Omega t)^2/2$ (here $a \ll \hbar/(m\omega)^{1/2}$) is in its **ground** state. Identify the perturbation Hamiltonian and, using the first order time-dependent perturbation theory.

(a) Ffind the probability $P_{0\to N}(t)$ of a transition from the ground ($N = 0$) to an arbitrary excited ($N = 1, 2, \ldots$) state near the resonance ($\Omega \approx \omega$);

(b) Find the probability of the inverse transition $P_{N\to 0}(t)$ if the system starts out in the N^{th} excited state;

(c) Find the probability $P_{0\to N}(t)$ if, instead of the oscillatory motion (as in a) and b)), the minimum of the potential $V(x,t) =$

$m\omega^2 x^2/2$ undergoes a **sudden** shift $(0 \rightarrow a)$ at $t = 0$ and then jumps back $(a \rightarrow 0)$ at time t.

QM2-4 Consider two spin 1/2 particles in a three-dimensional box of size L, $V(x, y, z) = 0$ *for* $0 < x < L,\ 0 < y < L,\ 0 < z < L$, otherwise $V = \infty$.

(a) If there is no interaction between particles, write down the expression for the two-particle ground state wave function (include both the spatial and spin parts, specifying all the quantum numbers). What is the energy and the degeneracy of this ground state?

(b) What is the energy and degeneracy of the first excited state?

(c) If there is a short-range repulsive interaction between the two particles, what would be the degeneracy of the first excited state?

QM2-5 Consider a soft spherical scattering potential $V(r) = V_0,\ for\ r <= R,\ and\ V(r) = 0,\ for\ r > R.$

(a) Using the first order Born approximation to calculate the scattering amplitude as a function of the scattering angle θ.

(b) Calculate the total scattering section in the low energy limit $kR << 1$. Under what condition is this cross section the same as that given by a hard sphere with the same radius?

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Astrophysics I: May 11, 9:00am-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject problem number and your PID, but not your name.

AS1-1 This question relates to continuum opacities in stars.

(a) List, and sketch the shapes of, the most important opacity sources in the atmospheres of late F and early G stars in the visible $(0.3 - 1\,\mu m)$ wavelength spectrum. Compute Boltzmann factors if necessary.

(b) Show how a discontinuity in the continuum spectrum in this wavelength interval can be used to measure n_e of these stars if T_{eff} can be obtained by other means.

AS1-2 Consider dimensional analysis of fully convective low-mass stars whose structural equations include

$$P = K_a \rho^{5/3}$$
$$T = K_b P^{2/5}$$

with K_i dimensionless constants for these stars.

(a) Use the stellar structure differential equations in Lagrangian form to show that the relation between radius and mass is $R_* \propto M^{-1/3}$

(b) From this show that the stellar mass/luminosity ratio $\Upsilon \propto M^{-2(1+2n/3)}$, with n the temperature exponent in the expression for the luminous flux increment per mass dL/dm on the lower main sequence.

AS1-3 A star of mass M has mass density profile that decreases from center to surface as $\rho(r) = \rho_c \left[1 - (\frac{r}{R})^2\right]$

(a) Show that $\rho_c = 15M/8\pi R^3$ and $P_c = \rho_c M/2R$

(b) Derive an expression to estimate the minimum stellar mass required for the central ignition of the various fuels with fusion threshold temperature T_c. Your final answer will be an inequality written using the degenerate electron degeneracy pressure $P_e = K(\rho_c/\mu_e)^{5/3}$ with K a constant and μ_e the electron mean molecular weight. Assume the density profile given in part (a), solar chemical composition μ, and non-degenerate electrons.

AS1-4 Use the Eddington approximation of constant specific intensity I into a hemisphere and a plane-parallel grey atmosphere in LTE to show that

(a) the angle averaged intensity of the radiation field at shallow optical depth τ is
$J(\tau) \propto T_{eff}^4(\tau + \frac{2}{3})$

(b) Hence that
$T^4(\tau) = \frac{3}{4}T_{eff}^4(\tau + \frac{2}{3})$
ie. the "surface" of an unresolved star of T_{eff} is at optical depth $\tau = \frac{2}{3}$.

AS1-5 Approximate a white dwarf of mass M and radius R as a degenerate core surrounded by an ideal gas atmosphere.

(a) Show that WD atmospheric pressure as function of temperature is in general
$P(T) = \left(\frac{64\pi acRG}{51\kappa_0\mu}\right)^{1/2}(\Upsilon)^{1/2}T^{14/4}$
for mass/luminosity ratio Υ.

(b) From this expression, show that the temperature profile throughout this atmosphere is
$T(r) = \frac{4}{17}\frac{\mu}{R}GM(\frac{1}{r} - \frac{1}{R})$

2009 Qualifying Exams

Department of Physics and Astronomy, UNC Chapel Hill

Astrophysics 2: May 11, 9:00-12:00pm

Choose 3 out of 5 problems

Start a new page for each problem. Label each page with the subject-problem number and your PID, but not your name. Note useful information on last page.

AS2-1. Suppose an extremely thin disk is embedded in a spherical dark-matter halo with constant density ρ_H. A spherical cloud of globular clusters orbits in the halo. Assume the halo strongly dominates the mass (i.e., the stellar mass is negligible).

a) Derive an expression for V(R) from force balance for stars on circular orbits, and use it to prove that the disk has the same angular speed at all radii. How does such a rotation curve compare to that of the Milky Way?

b) Prove that a globular cluster on a *radial* orbit (different force equation!) will execute simple harmonic motion, and derive the harmonic frequency.

c) Show that the self-potential energy of this system is $\Phi\ (R) = \frac{16}{15}\pi^2 G\rho_H^2 R^5$.

d) If ρ_H is unknown, how can Φ be combined with the globular cluster velocity dispersion to estimate the mass of the system from the Virial Theorem? You may assume that we are only interested in the mass within the radius of the clusters. Explain what, if any, additional assumptions would be needed, and set up the calculation so that only algebra is left.

AS2-2. A dwarf galaxy is merging into a much larger primary galaxy, with current separation 50 kpc. The dwarf has total mass $5x10^8$ M_{sun} and the primary has stellar mass $3x10^{10}$ M_{sun}. The primary contains $5x10^9$ M_{sun} in gas as well, of which half is currently in molecular form with n_H~10^4-10^6 cm^{-3} and located at central radii <5 kpc. The atomic gas extends to about 20 kpc and its rotation curve implies a total mass of 10^{11} M_{sun} for the primary. The dwarf has a few loosely bound globular clusters of its own. Assume that the system evolves as a closed box, and the current star formation rate is 10 M_{sun}/yr.

a) What will be the order within each pair of events listed below? Justify your answers both in words and based on one or more equations from the last page, without actually doing any algebra or calculations.

i) explosion of stars that increase [α/Fe] *vs.* emergence of line blanketing shortward of 4000 Angstroms in the integrated galaxy spectrum

ii) collapse of the molecular gas *vs.* infall of the atomic gas from 20 kpc to replenish the molecular gas

iii) merging of the dwarf galaxy into the primary *vs.* merging of the globular clusters into the primary (assume that due to tidal truncation and disk shocking, the globular clusters take a separate inspiral trajectory from their former host galaxy)

b) Assuming all of the gas falls in, gets converted into molecular gas, and collapses quickly enough to feed star formation continuously with $SFR \propto M_{gas}$, prove that the gas mass drops off exponentially and find the value of the $1/e$ timescale. (Note that even if made of pure gas, the dwarf would have 0.1x the gas mass of the primary, so we may ignore its contribution for this calculation.)

AS2-3. Consider a "singular isothermal sphere" (SIS), with density profile $\rho_{SIS} = \frac{1}{4\pi G}\frac{V_{SIS}^2}{R^2}$. Ignore the singularity at the origin when answering the questions below. (People often study "lowered isothermal profiles" in which this singularity is replaced with something finite.)

a) Integrate the Poisson equation to show that the potential could be $\varphi = V_{SIS}^2 \ln(R)$. Why is the potential not unique?

b) Consider an ultra-thin disk embedded in a dark-matter halo with SIS density profile. Assume you can neglect z motion and just examine orbits in cylindrical R,θ coordinates, expressing circular velocity as V_θ. Write down the force equation for $\ddot{R}$, and find an expression for $\varphi_{eff}(R, L_z)$ that allows you to rewrite the force equation as $\ddot{R} = -\frac{\partial \varphi_{eff}}{\partial R}$.

c) Find an expression for the rotation curve V(R) of the disk, equivalent to the guiding center orbit velocities as a function of radius. What would you expect for the approximate radius of a bar with pattern speed Ω_p in this disk?

d) If a star gains a small radial velocity, it oscillates in epicycles around its guiding center orbit. What about the effective potential causes this behavior?

AS2-4. We have seen that the Faber-Jackson relation can be derived by assuming constant M/L and constant surface brightness along with the Virial Theorem. Taking a more general approach, we can derive the Fundamental Plane simply from the definition $I_e = \frac{1}{2}\frac{L}{\pi R_e^2}$ and the assumption that E galaxies are "homologous," meaning $KE = \frac{3}{2}c_1 M\sigma_e^2$ and $PE = -\frac{c_2 M^2}{R_e}$ with the same constants c_1 and c_2 for all galaxies. These constants contain the details of specific density profiles.

a) Show that these equations imply that $\log R_e = 2\log\sigma_e - \log\frac{M}{L} - \log I_e + c$ (which is very similar to the observed Fundamental Plane, when corrected for M/L variations).

b) The galaxies described below do not fall on the z=0 Fundamental Plane, even after correcting for stellar population variations. Does this mean that they are not in virial equilibrium? Why might the authors think that dry merging can help move these galaxies to the z=0 FP? Why wouldn't wet merging work?

CONFIRMATION OF THE REMARKABLE COMPACTNESS OF MASSIVE QUIESCENT GALAXIES AT $Z \sim 2.3$: EARLY-TYPE GALAXIES DID NOT FORM IN A SIMPLE MONOLITHIC COLLAPSE [1,2]

PIETER G. VAN DOKKUM[3], MARIJN FRANX[4], MARISKA KRIEK[5], BRADFORD HOLDEN[6], GARTH D. ILLINGWORTH[6], DANIEL MAGEE[6], RYCHARD BOUWENS[6], DANILO MARCHESINI[3], RYAN QUADRI[4], GREG RUDNICK[7], EDWARD N. TAYLOR[4], AND SUNE TOFT[8]

Accepted for publication in ApJ Letters

ABSTRACT

Using deep near-infrared spectroscopy Kriek et al. (2006) found that $\sim 45\%$ of massive galaxies at $z \sim 2.3$ have evolved stellar populations and little or no ongoing star formation. Here we determine the sizes of these quiescent galaxies using deep, high-resolution images obtained with HST/NIC2 and laser guide star-assisted Keck/AO. Considering that their median stellar mass is $1.7 \times 10^{11} M_\odot$ the galaxies are remarkably small, with a median effective radius $r_e = 0.9$ kpc. Galaxies of similar mass in the nearby Universe have sizes of ≈ 5 kpc and average stellar densities which are two orders of magnitude lower than the $z \sim 2.3$ galaxies. These results extend earlier work at $z \sim 1.5$ and confirm previous studies at $z > 2$ which lacked spectroscopic redshifts and imaging of sufficient resolution to resolve the galaxies. Our findings demonstrate that fully assembled early-type galaxies make up at most $\sim 10\%$ of the population of K-selected quiescent galaxies at $z \sim 2.3$, effectively ruling out simple monolithic models for their formation. The galaxies must evolve significantly after $z \sim 2.3$, through dry mergers or other processes, consistent with predictions from hierarchical models.
Subject headings: cosmology: observations — galaxies: evolution — galaxies: formation

c) One projection of the FP is $\frac{SB_e}{mag\ arcsec^{-2}} = 2\log\left(\frac{R_e}{kpc}\right) + const$, which is a version of the Kormendy Relation. Here $SB_e = -2.5\log I_e + const$. This relation is sometimes used as a crude distance indicator. Explain mathematically/geometrically why one of these variables is independent of distance in the local universe. Given this argument, how can you use the relation as a distance indicator? Describe the optimal method of fitting the calibrating data that defines the relation (for which distances are known from another source) – which variable should you minimize scatter in? Give at least two reasons why this distance indicator will not work well at high redshift.

AS2-5. The table below summarizes much of what we know about different components of the Milky Way disk, and the spectrum below the table shows the Lyman alpha forest spectrum for a distant quasar (both figures taken from Sparke & Gallagher).

Table 2.1 Scale heights and velocities of gas and stars in the disk and halo

Galactic component	h_z *or shape*	$\sigma_x = \sigma_R$ (km s^{-1})	$\sigma_y = \sigma_\phi$ (km s^{-1})	σ_z (km s^{-1})	$\langle v_y \rangle$ (km s^{-1})	*Fraction of local stars*
HI gas near the Sun	130 pc		≈5	≈7	Tiny	
Local CO, H_2 gas	65 pc		4		Tiny	
Thin disk: $Z > Z_\odot/4$	(Figure 2.9)					90%
$\tau < 3$ Gyr	≈280 pc	27	17	13	−10	
$3 < \tau < 6$ Gyr	≈300 pc	32	23	19	−12	
$6 < \tau < 10$ Gyr	≈350 pc	42	24	21	−19	
$\tau > 10$ Gyr		45	28	23	−30	
Thick disk	0.75–1 kpc					5%–15%
$\tau > 7$ Gyr, $Z < Z_\odot/4$	(Figure 2.9)	68	40	32	−32	
$0.2 \lesssim Z/Z_\odot \lesssim 0.6$		63	39	39	−51	
Halo stars near Sun	$b/a \approx 0.5$–0.8					~0.1%
$Z \lesssim Z_\odot/50$		140	105	95	−190	
Halo at $R \sim 25$ kpc	Round	100	100	100	−215	

Note: gas velocities are measured looking up out of the disk (σ_z of HI), or at the tangent point (σ_ϕ for HI and CO); velocities for thin-disk stars refer to Figure 2.9. For thick disk and halo, abundance Z, shape, and velocities refer to particular samples of stars. Velocity $\langle v_y \rangle$ is in the direction of Galactic rotation, relative to the *local standard of rest*, a circular orbit at the Sun's radius R_0, assuming $v_{y,\odot} = 5.2$ km s^{-1}.

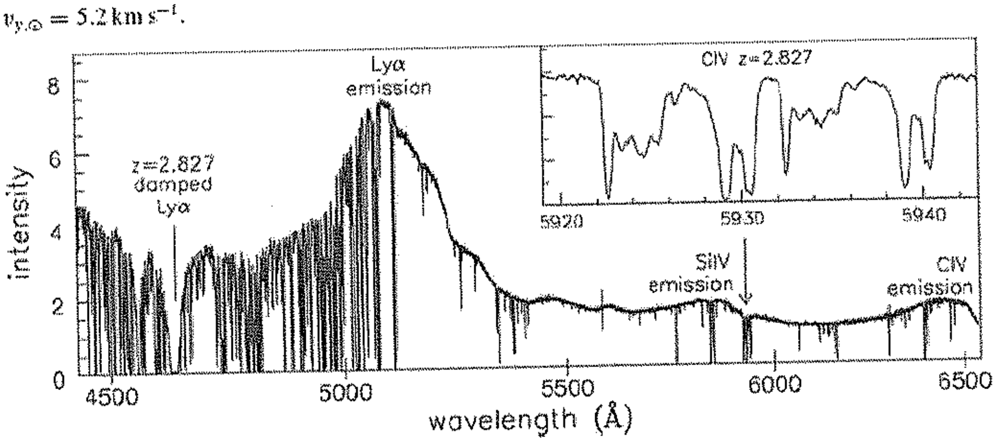


Fig. 9.12. The spectrum of quasar 1425 + 6039 with $z_{em} = 3.173$: broad Lyα emission at 1216 Å is redshifted to the visible region. At shorter wavelengths, narrow absorption lines of the Lyα forest are dense. The squarish profile at 4650 Å is a damped line of Lyα, at $z_{abs} = 2.827$. The arrow shows absorption at the same redshift in the CIV doublet with rest wavelength near 1550 Å: the inset reveals distinct absorption components from multiple gas clouds – L. Lu and M. Rauch.

a) HI outer disks of galaxies typically have N_{HI} = a few x 10^{20} cm^{-2}, equivalent to a few M_{sun}/pc^2, and are believed to be responsible for "damped Lyman α" absorption such as that shown in the QAL spectrum. UV radiation only partially ionizes the HI in a skin, analogous to the boundary between HII regions and HI/H_2 gas clouds. Assume that DLAs are similar to the Milky Way, and that atoms recombine as fast as UV photons can ionize them. Roughly how thick is the partially ionized skin compared to the thickness of the HI disk? Does the quasar-ionized layer reach the molecular gas? You may approximate the gas distribution as having constant column density in each dz-height interval within a range $\pm h_z$ from the center of the disk plane, and zero beyond $\pm h_z$ (top-hat rather than exponential). *Hint: see useful info on last page.*

b) Compare the vertical scale height of the molecular gas with the Jeans length for this gas component. Explain the (order-of-magnitude) physical reasoning for the form of the equation for the Jeans length. You may assume a molecular gas density of 10^4 cm^{-2}. Discuss the results based on your expectations for a gas layer in equilibrium.

c) The table indicates increasing scale heights and velocity dispersions for stars of different ages τ compared to the gas layer. Describe the physical process that causes this effect. What does the tensor virial theorem have to do with the change in scale height? Why is $\langle v_y \rangle$ becoming more negative as velocity dispersion increases? (See definition of $\langle v_y \rangle$ in caption.)

d) According to the table caption, the Sun has positive $\langle v_y \rangle$. Is the guiding center of its orbit at larger or smaller galactocentric radius than our current position? Explain based on angular momentum conservation.

Possibly Useful Facts & Equations

$H_0 = 70\ km\ s^{-1}\ Mpc^{-1}$ Hubble constant

$G = 4.28x10^{-6} kpc \cdot km^2 \cdot \sec^{-2} \cdot M_{sun}^{-1}$

$c\ =\ 3x10^5\ km/sec$

$\sigma_H = 10^{-17} cm^2$ ionization cross section of neutral Hydrogen to ionizing photons

$\lambda = 2.9\ mm \cdot K/T$ Wien's displacement law

$2KE + PE = 0$ Virial Theorem

$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial\varphi}{\partial R}\right) = 4\pi G\rho$ Poisson Equation for a spherically symmetric mass distribution

$Z(t) = -p \ln\left(\frac{M_g(t)}{M_g(0)}\right)$ equation for metallicity evolution with time

$\tau_{MS} = 10^{10}\left(\frac{M}{M_{sun}}\right)^{-2.5} yr$ main sequence lifetime of a star of mass M

$t_{free-fall} = \sqrt{\frac{3\pi}{32G\rho}} = 5x10^7 yr\ (n_H\ in\ cm^{-3})^{-0.5}$ free-fall time for pressureless gas cloud of initial density ρ

$t_{dyn} = \sqrt{2}\ t_{free-fall}$ dynamical time for a test particle to fall to the middle of a constant-density potential

$t_{relax} = \frac{0.1N}{\ln N}\frac{R}{V} = 10^6 yr \frac{0.1N}{\ln N}\frac{R\ in\ pc}{V\ in\ km/s}$ relaxation time due to weak star-star interactions

$t_{fric} = \frac{2.64\ x\ 10^{11}}{\ln\Lambda}\left(\frac{r_i}{2\ kpc}\right)^2\left(\frac{v_c}{250\frac{km}{s}}\right)\left(\frac{10^6 M_{sun}}{M}\right) yr$ dynamical friction timescale

$\frac{dV}{dt} \propto -M\rho/V^2$ Chandrasekhar formula for dynamical friction on a satellite of mass M

$\log\left(\frac{M_{BH}}{10^8 M_{sun}}\right) = 4\log\left(\frac{\sigma_{los}}{200\frac{km}{s}}\right) + 0.2$ approximate black-hole mass vs. bulge σ relation

$\kappa^2 = (R\frac{\partial\Omega^2}{\partial R} + 4\Omega^2)$ epicyclic frequency in relation to angular rotation speed in a thin disk

And recall that within astronomical accuracy, $100km/s \times 1Gyr = 100kpc$

QUALIFYING EXAM 2008

MAY 09 2008, PART 1

PID :...

NAME :..

There are 3 sections in Part 1: Classical Mechanics, Electromagnetism I, Quantum Mechanics 1. There are 5 problems in each section. You have 1.5 hrs available for each section, a total of 4.5 hrs. Write your name only on this coversheet and not in any other pages.

Please present your solutions in the sheets you brought. You are allowed calculators and scrap paper. Write your PID (but not the name) in everv page! Number the pages for each problem includina the total number of pages used for that problem, e.g. "Problem2-QM1,3/5" means the third page out of 5 pages that you used for Problem2 of QM1. This will ensure that no pages are missing from your work.
Good Luck!

EMI: NO. SCORE

1..
2..
3..
4..
5..

QMI: NO. SCORE

1..
2..
3..
4..
5..

CM: NO. SCORE

1..
2..
3..
4..
5..

Problem 1 Mechanics

1. Consider a ball of mass m dropped and energy bouncing elastically up and down (so that energy is conserved).

   (a) Sketch the trajectories in phase space

   (b) Express the Hamiltonian in terms of the action variable J and calculate the frequency of oscillation in terms of E.

Problem 3 Mechanics

3. Consider the "point" transformation from the coordinates $q_1 \ldots q_N$ to another set $x_1 \ldots x_n$. Show that if Lagrange's equations hold for the q's they also hold for the x's provided the functions $x_i(q_1, \cdots, q_N, t)$, $i = 1, N$ satisfy a certain mathematical condition. What is the condition and what does it mean physically?

Problem 4 Mechanics

A particle of mass m is subject to a central force $F(r) = -V'(r)$. Assume the particle moves on a circular orbit of radius $r = R$, and that in that orbit its angular momentum is L.

Assume very little specific information about the potential away from $r = R$ except that it has a Taylor series expansion

$$V(r) = V(R) + V'(R)\ (r - R) + \frac{1}{2}V''(R)(r - R)^2 + \cdots.$$

(a) Determine the angular momentum L, energy E, and angular velocity Ω_ϕ of the circular orbit.

(b) Consider a nearly circular orbit with the same angular momentum. Work out the linear perturbation equation for radial motion.

(c) What condition on the derivative $V'(R)$ and second derivative $V''(R)$ must hold for the perturbed orbit to be stable?

(d) After finding the frequency Ω_r of radial motion, give an expression for the change in apsidal angle $\Delta\phi$ that occurs per radial oscillation.

Problem 5 Mechanics

The motion of a relativistic particle of mass m in a static potential $V(\vec{x})$ can be obtained from the Lagrangian

$$L = -mc^2(1 - v^2/c^2)^{1/2} - V(\vec{x}).$$

(a) Write out Lagrange's equations.

(b) Find the canonical momentum $\vec{p}$ and write out the Hamiltonian $H(x^i, p_i)$ (in terms of position and momentum).

(c) Is H a constant of the motion?

Statistical Mechanics

2008

Problem 1 SM

Consider a 3 dimensional quantum solid of N weakly-interacting, distinguishable particles. Each particle experiences a harmonic oscillator potential given by $V(r) = \frac{1}{2}kr^2$, where r is the distance of the particle from its location in the solid. Find expressions for the entropy, the total internal energy, and the heat capacity as functions of temperature.

Problem 2 SM

Consider a system of identical, distinguishable particles where the energy-levels of each particle are quantized as follows:

$$E_n = E_0 \ln(n); \;\; n = 1, 2, 3, 4, ...$$

1. Derive an expression for the probability of finding a particle in the $n = 2$ state relative to the probability of finding the particle in the $n = 1$ state as a function of temperature.
2. Find the numeric value for this relative probability for the special case when $kT = E_0$.
3. Find the entropy of the system as a function of temperature. Assume that the maximum energy of each particle is limited to $n = 3$.

Statistical Mechanics

2008

Problem 3 SM

Consider a system of N indistinguishable classical ultrarelativistic particles confined in volume V. The temperature of this system is given, T. The Hamiltonian of such ultrarelativistic particles is given by

$$H(\vec{q}_i, \vec{p}_i) = \sum_{i=1}^{N} |\vec{p}_i| c = \sum_{i=1}^{N} c\sqrt{p_{ix}^2 + p_{iy}^2 + p_{iz}^2}$$

(1) Show that the free energy is given by

$$F(T,V,N) = -NkT\left[1 + \ln\left\{\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right\}\right]$$

(2) Show that $pV = NkT$ is valid for this system.

(3) For given chemical potential μ, show that the grand potential, $\phi(T,V,\mu) = U - TS - \mu\langle N\rangle$, of this system is given by $\phi = -kTe^{\mu/kT} 8\pi V\left(\frac{kT}{hc}\right)^3$.

Problem 4 SM

A surface with N_0 adsorption centers has $N\,(<N_0)$ gas molecules adsorbed on it. Show that the chemical potential of the adsorbed molecules is given by

$$\mu = kT\ln\frac{N}{(N_0 - N)a(T)}$$

where $a(T)$ is the partition function of a single adsorbed molecule. Neglect the intermolecular interaction among the adsorbed molecules (thus, it works okay for Ar adsorption but not for H_2O on surfaces). Assume N, N_0, and N_0-N are all very large numbers.

Problem 5 SM

Consider the Ising model of N magnetic spins. The Hamiltonian is given by

$$H = -\sum_{i=1}^{N} B\mu\sigma_i - \frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j; \quad \sigma_i, \sigma_j = \pm 1;$$

$$J_{ij} = J, \ i \text{ and } j \text{ nearest neighbors}$$
$$= 0, \text{ otherwise}$$

where B is the external magnetic field, μ is the magnetic moment, and J is the interaction strength between two nearest-neighbor spins. Use the mean-field approximation to calculate the partition function.

(1) Show that the magnetization $M = N\mu\langle\sigma\rangle$ is given by

$$M = N\mu \tanh\left[\frac{\mu}{k_B T}\left(\frac{qJ}{\mu}\langle\sigma\rangle + B\right)\right]$$

where q is the number of the nearest neighbors.

(2) Consider B=0. Show that the critical temperature $T_c = \frac{qJ}{k_B}$.

(3) Show that near the critical temperature and $T < T_c$, $\langle\sigma\rangle \propto \left(1 - \frac{T}{T_c}\right)^{\beta}$, and the critical exponent $\beta = 1/2$.

Remember that $\tanh x = x - \frac{1}{3}x^3 + \cdots$

Quantum Mechanics - II

2008

Problem 1 QM-2

A particle of mass m is confined by the two-dimensional oscillator potential $V(x,y) = m\omega^2(x^2+y^2)/2$ and subject to a time-independent perturbation $\delta V(x,y) = \alpha xy$ where $\alpha \ll m\omega^2$.

a) Find the ground state energy $E_{00} = E_{00}^{(0)} + E_{00}^{(1)} + E_{00}^{(2)} + \ldots$ to second order in α and the ground state wavefunction $\psi_{00}(x,y) = \psi_{00}^{(0)}(x,y) + \psi_{00}^{(1)}(x,y) + \ldots$ to first order.

b) Find the energy splitting $E^{(\pm)}$ and the corresponding "good" linear combinations $\psi^{(\pm)}(x,y)$ of the lowest excited states $\psi_{01}(x,y)$ and $\psi_{10}(x,y)$ which would be degenerate in the absence of the perturbation ($E_{10}^{(0)} = E_{01}^{(0)}$).

Problem 2 QM-2

A particle of mass m is confined in a one-dimensional infinite square well of width a and subject to a weak time-dependent perturbation $V(x,t) = \alpha\cos(\pi x/a)\cos\omega t$. At $t=0$ the system is in the ground state.

a) Find (in the second order in α) the probability $P_{1\to n}(t)$ of a transition from the ground ($n=1$) to an arbitrary ($n = 2, 3, \ldots$) excited state provided that $\hbar\omega \approx 3\pi^2\hbar^2/2ma^2$.

b) What is the probability of the inverse transition $P_{n\to 1}(t)$ if the system starts out in the n^{th} excited state?

c) What is the probability $P_{1\to n}(T)$ in the case of the perturbation $V(x,t) = \alpha\cos(\pi x/a)\theta(t)\theta(T-t)$?

here $\theta(t) = 1$ is the step-function: $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$.

Problem 3 QM-2

Consider two spin-1/2 fermions of mass m in a 1-d box of length L.

1. Start off by letting only one particle be in the box. Using periodic boundary conditions, write down the spatial part of the lowest-energy wave function and the corresponding energy (which does not depend on the spin). Also write down the spatial parts of the next two lowest wave functions (they are degenerate) and the corresponding energy.

2. Now both particles are in the box. Caluclate the first-order energy shift of the lowest spin-triplet state when the two particles interact through a two-body potential $V = K\delta(x_1 - x_2)$, where K is a constant. What physical phenomenon is reflected in your result?

3. Calculate the first-order shift for the lowest spin-singlet state.

Electromagnetism –II

2008

Problem 1 – EM II

A single harmonically oscillating dipole, with dipole moment $\vec{d}(t) = \vec{d}_0 e^{-i\omega t}$, that is centered on the origin gives rise to a complex electric field amplitude

$$\vec{E}(\vec{x})_{\text{dipole}} = -k^2 \frac{e^{ikr}}{r} \vec{n} \times (\vec{n} \times \vec{d}_0)$$

in the radiation zone $kr \gg 1$.

Assume the existence of two such dipoles, which are both oscillating at the same frequency. Assume that one dipole is displaced from the origin to a position $\vec{x} = \vec{x}_0$ and the other is displaced to $\vec{x} = -\vec{x}_0$. Assume the displacements are along the z axis and that $k|\vec{x}_0| \ll 1$. Let the dipole moments be related by $\vec{d}_1 = \vec{d}_2 = \vec{d}_0$ (i.e., they are in phase with each other) and take the dipole vector to lie along the z axis also.

1. In phase displaced dipoles give rise to a dipole field plus corrections. Obtain an expression for the electric field in the radiation zone of the combined source that is accurate through the first two nonvanishing orders in k.

2. Let θ be the polar angle relative to the z axis. Compute the angular distribution of radiated power, $dP/d\Omega$, through the first two nonvanishing orders in k. Sketch the angular distribution of radiated power of the correction term, describe the multipole, and why it arises.

Problem 4 – EM II

For a relativistic electron the time-dependent angular distribution of radiated power is

$$\frac{dP}{d\Omega}(t') = \frac{e^2}{4\pi c} \frac{|\vec{n} \times \{(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}\}|^2}{(1 - \vec{n} \cdot \vec{\beta})^5},$$

as a function of the charge's time parameter t' (i.e., $\vec{\beta} = \vec{\beta}(t')$ and $\dot{\vec{\beta}} = \dot{\vec{\beta}}(t')$).

Consider an electron that is only moving and accelerating along the z-axis. Let θ be the polar angle of the observer relative to the z axis. Assume that the electron is executing simple harmonic motion, $z(t') = a \cos \omega_0 t'$, with a fixed frequency ω_0 and amplitude a. Define a parameter $\beta_0 = a\omega_0/c$; then $\beta(t') = -\beta_0 \sin \omega_0 t'$. Assume that $\beta_0 \ll 1$.

1. Find the first three or more terms in an expansion in powers of β_0 of the time-dependent radiated power.
2. Time average the terms in the power series. Give the angular distribution of power radiated in the first two multipoles (i.e., dipole and quadrupole).
3. Integrate for the total power radiated in the first two multipoles.

QUALIFYING EXAM 2008

May 12 2008, Astronomy PART 2

PID : ....

Name: ....

....

There are 3 sections in Astro-track: Statistical Mechanics, Astro-I and Astro-II (Galaxy). There are 5 problems in each section. You have 1.5 hrs available for each section, a total of 4.5 hrs.

Please present your solutions in the sheets you brought. You are allowed calculators and scrap paper. Write vour PID (but not your name) in everv page! Your name is on the cover page of the exam only. Staple sheets of each problem individually. Number the pages for each problem including the total number of pages used for that problem, e.g. "Problem2-QM2,3/S9 means the third page out of 5 pages that you used for Problem2 of QM2.This will ensure that no pages are missing from your work.

* ***THIS PART OF THE EXAM IS FOR ASTRONOMY TRACK ONLY.*** *

Good Luck!

SM:	**NO.**	**SCORE**
1.	....	....
2.	....	....
3.	....	....
4.	....	....
5	....	....

ASTRO1 (STELLAR PHYSICS) :	**NO.**	**SCORE**
1.	....	....
2.	....	....
3.	....	....
4.	....	....
5.	....	....

ASTRO 2 (GALAXY) :	**NO.**	**SCORE**
1.	....	....
2.	....	....
3.	....	....
4.	....	....
5.	....	....

A1

Problem 1

- Write down the 4 equations of stellar structure in both Eulerian and Lagrangian differential forms. What constitutive physics (e.g. equation of state) couples the 4 equations together, requiring simultaneous solution? Which equations decouple for equations of state depending only upon density? Use this fact to write down a general mass-radius relationship for bodies where $P \propto \rho^\nu$.

A1

Problem 3

- Sketch a typical curve of growth for a spectral line and explain why it has three differently sloped regions. What mechanism dominates line growth in each region, and how does this mechanism depend on atmospheric properties such as temperature and chemical abundance?

A1

Problem 5

- Estimate the hydrogen burning lifetimes of stars on the lower and upper ends of the main sequence. The lower end occurs near 0.085 $M_\odot$ with $\log_{10}(L/L_\odot) = -3.297$ & $\log_{10} T_{eff} = 3.438$, and the upper end near 90 $M_\odot$ with $\log_{10}(L/L_\odot) = 6.045$ & $\log_{10} T_{eff} = 4.722$. If you think that either of these stars will be completely convective on the main sequence, be sure to use the entire mass of hydrogen for fuel not just the inner 10% in the fusing core.

1. Black Holes and Galaxy Dynamics

(a) Assuming a dominant dark matter halo (hence roughly spherical mass distribution), combine expressions for the centripetal and gravitational forces on a star to derive a general formula for a galaxy's enclosed dynamical mass as a function of radius r and rotation velocity V.

(b) The large-scale rotation curve of a galaxy obeys $V(r) = 200\ km/s\left(\frac{r}{1\ kpc}\right)$ up to 1 kpc and stays flat at 200 km/s from 1–10 kpc. However, on *nuclear* scales a pair of black holes with total mass $10^7\ M_{sun}$ orbit each other and influence the galaxy's dynamics. At small radii where the influence of the black hole pair dominates and the large-scale rotation is negligible, what is the expected shape of the rotation curve as a function of r?

(c) Find the radius bounding the black hole pair's "sphere of influence," defined as the radius where the gravitational force from the black hole pair and the galaxy are equal. You may use $G=4.28 \times 10^{-6}\ kpc*km^{2}*sec^{-2}*M_{sun}^{-1}$.

(d) Describe the effect of close interactions between individual stars and the black hole pair on the orbits of each. How will many such interactions affect the galaxy light profile?

3. The Tilt of the Faber-Jackson Relation

We have seen that the Faber-Jackson relation for spheroids may be approximately derived from the Virial Theorem by assuming that all spheroids share the same mass-to-light ratio γ and mean surface brightness Σ. Suppose we relax these assumptions and assume only that M/L increases as a power law in L, i.e., $\gamma \propto L^{\alpha}$.

(a) Starting from the Virial Theorem, show that $L \propto \sigma^{4/(1+2\alpha)}$.

(b) Assuming that massive elliptical galaxies are dominated by baryonic matter within their visible extent, and considering what you know about the stellar populations of massive elliptical galaxies, explain which way the relation should tilt, or equivalently, what should be the sign of α if L is measured in the B band. How will the tilt change in redder passbands?

(c) Suppose instead that dark matter fraction increases as a function of stellar mass for elliptical galaxies, and stellar population differences are negligible. How does γ depend on L now? How could you observationally distinguish this scenario from the changing stellar populations scenario?

(d) Which variable is measured and which inferred when the Faber-Jackson relation is used as a distance indicator? How do the tilts discussed above affect the predictive power of the relation as a distance indicator, assuming that a significant fraction of the scatter is in σ?

5. Mass Determination via the Virial Theorem

A cluster of galaxies follows a Plummer sphere potential with $b = 2$ Mpc and mean line-of-sight velocity dispersion σ=1000 km/sec.

$$\Phi_P = -GM/\sqrt{r^2 + b^2}$$

(a) Without using math, explain how you know that the total mass of this cluster is M.

(b) Use the Poisson Equation to show that the cluster mass density $\rho = \frac{3M}{4\pi}\frac{b^2}{(r^2+b^2)^{\frac{5}{2}}}$.

Recall that $\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}\Phi)$.

(c) The total potential energy of the cluster is $PE = 1/2\int \rho\Phi dV = -\frac{3\pi}{32}GM^2/b$. From the Virial Theorem, estimate the mass M of the cluster. You may use G=4.28 x 10^{-6} kpc*km^2*sec^{-2}*M_{sun}^{-1}.

(d) What characteristic of this cluster seems to justify the use of the Virial Theorem?

Numerical Constants:

Solar Mass (Msun):	1.989×10^{33} g
Solar Radius (Rsun):	6.96×10^{10} cm
Solar Luminosity:	3.847×10^{33} erg/s
Gravitational Constant (G)	6.6726×10^{-8} cm^3/g/s^2
Proton mass	1.6726×10^{-24} g = 938.27 MeV/c^2
Yield of p-p reactions (Q)	26.7 MeV = 4.28×10^{-5} ergs

Numerical Constants:

Solar Mass (Msun):	1.989×10^{33} g
Solar Radius (Rsun):	6.96×10^{10} cm
Solar Luminosity:	3.847×10^{33} erg/s
Gravitational Constant (G)	6.6726×10^{-8} cm^3/g/s^2
Proton mass	1.6726×10^{-24} g = 938.27 MeV/c^2
Yield of p-p reactions (Q)	26.7 MeV = 4.28×10^{-5} ergs

Boltzmann constant	1.38×10^{-16} erg/K
Planck's Constant	6.626×10^{-27} erg-s
Electron mass	9.109×10^{-28} g

$m_p = 1.6726 \times 10^{-24}$ g

Appendix A (The 2007 Qualifier)

NOTE: The 2007 Qualifier questions have been copied word-for-word as they appeared on the qualifier, including grammatical errors and misspellings.

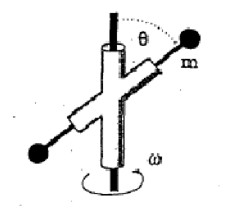
CM-1

Consider a ball of mass *m* dropped and energy *E* bouncing elastically up and down (so that energy is conserved).

(a) Sketch the trajectories in phase space
(b) Express the Hamiltonian in terms of the action variable *J* and calculate the frequency of oscillation in terms of *E*.

CM-2

Consider a massless rigid rod of length *l* with a ball of mass *m* at each end, rotating around an axis that runs through the center of mass as shown (θ <90˚). The radius of each ball is negligibly small.



(a) What are the principal moments of inertia I_i in the body-fixed frame?
(b) The components of *w* are constant in the body-fixed frame. Find the components of *L* in that frame, and draw the direction of *L*.
(c) Use Euler's equations to find the direction of the torque *N* (in the body-fixed frame) required to keep the object rotating as in the figure. Draw the direction of *N*.

CM-3

Consider the "point" transformation from the coordinates $q_1 ... q_N$ to another set $x_1 ... x_n$. Show that if Langrage's equations hold for the *q*'s they also hold for the *x*'s provided the functions $x_i(q_1 ... q_N, t)$, $i=1, N$ satisfy a certain mathematical condition. What is the condition and what does it mean physically?

CM-4

A particle of mass *m* is subject to a central force $F(r) = -V'(r)$. Assume the particle moves on a circular orbit of radius $r=R$, and that in that orbit its angular momentum is *L*.

Assume very little specific information about the potential away from $r=R$ except that is has a Taylor series expansion

$$V(r) = V(r) + V'(r)(r-R) + \frac{1}{2}V''(R)(r-R)^2 + ...$$

(a) Determine the angular momentum *L*, energy *E*, and angular velocity Ω_ϕ of the circular orbit.

(b) Consider a nearly circular orbit with the same angular momentum. Work out the linear perturbation equation for radial motion.
(c) What condition on the derivative *V'(R)* and second derivative *V''(R)* must hold for the perturbed orbit to be stable?

(d) After finding the frequency Ω_r of radial motion, give an expression for the change in apsidal angle $\Delta\phi$ that occurs per radial oscillation.

CM-5

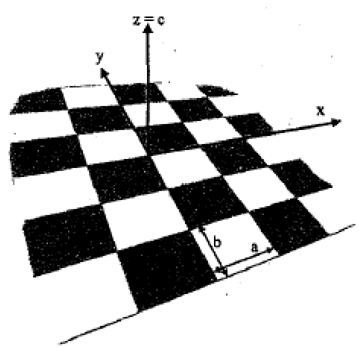
The motion of a relativistic particle of mass *m* in a static potential *V(x)* can be obtained from the Lagrangian

$$L = -mc^2(1-\frac{v^2}{c^2})^{\frac{1}{2}} - V(\mathbf{x}).$$

(a) Write out Lagrange's Equations.
(b) Find the canonical momentum *p* and write out the Hamiltonian $H(x^i,p_i)$(in terms of position and momentum).
(c) Is *H* a constant of the motion?

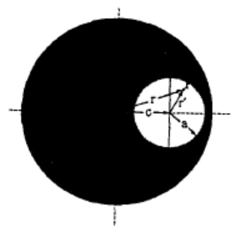
EM I – 1

The figure below shows an infinite checkerboard in the x-y plane in which the grey boxes are held at a potential $+V_o$ and the white boxes are at $-V_o$. The boxes have dimensions a x b as shown. Find the potential at all points z > 0, assuming the *z=c* plane is held at zero potential.



EM I – 2

A straight, cylindrical conductor carries a constant current density*J*. A cylindrical cavity of radius *a* is cut into the conductor, along an axis parallel to that of the conductor and offset by a distance *c*. A cross-sectional view is shown below. Assuming *J* is directed into the page, what is the magnitude and direction of the magnetic field at a point *P* within the cavity?



EM I – 3

Consider a uniformly charged sphere of radius *2a* and charge density ρ . Assume the sphere contains a spherical cavity or radius a/2 that is centered at (0, 0, 3*a*/2), while the larger sphere is centered at the origin. Find the force on a point charge of charge *q* located at (0, 0, *a*).

EM I – 4

Consider a point charge of charge *q* that is located at a height *h* above a large pool filled with a perfectly conducting fluid that has a mass density ρ . The pool is located at the surface of the earth. Assume that the deviation of the surface caused by the electrostatic force from the charge is much smaller than *h*.

1. Find the electric field at the surface of the fluid.
2. The electrostatic force per unit area experienced by a surface charge density at a surface where the electric field is discontinuous is given by:

$$\vec{f} = \sigma(\vec{E_1} + \vec{E_2})/2$$

where $\vec{E_1}$ and $\vec{E_2}$ are the electric fields at the two sides of the surface. Given this, find the equation of the surface the fluid assumes under the electrostatic force of the charge.

EM I – 5

A conductor at potential *V=0* has the shape of an infinite plane except for a hemispherical bulge of radius *a*. A charge *q* is placed above the center of the bulge, a distance *p* from the plane (or *p – a* from the top of the bulge). What is the force on the charge?

QM I – 1

Consider the coherent state of a one-dimensional simple harmonic oscillator $|\lambda\rangle = e^{-\lambda^2/2} e^{\lambda a^+} |0\rangle$, where λ is a number and a^+ is the creation operator. (a) Show that the coherent state satisfy the minimum uncertainty product for *x* and *p*. (b) Do an order-or-magnitude estimate for the value of λ for a macroscopic pendulum oscillator with string length of 1m, ball mass of 1kg, and oscillation amplitude of 10 degrees.

QM I – 2

Answer these questions briefly:

(a) Write down the relationship between the wave function in the coordinate space and that in the momentum space.
(b) Describe briefly what is the Aharnorov-Bohm effect.
(c) Write down the Wigner-Eckart theorem.
(d) Describe briefly the experiment that demonstrates the gravity-induced quantum interference effect.
(e) Derive the equation of motion for the time evolution of the density operator.

QM I – 3

Consider a beam of spin ½ particles in the pure state $|n,+\rangle$, where n is a unit vector with polar angle θ and azimuthal angle ϕ. Use the eigenvector of S_x, $|+\rangle$, $|-\rangle$ as the basis.

(a) Show explicitly that $|n,+\rangle = \cos(\theta/2)|+\rangle + e^{i\phi}\sin(\theta/2)|-\rangle$ is the eigenstate of $S_n = S \cdot n$ with the eigenvalue $\hbar/2$?
(b) A S_y Stern-Gerlach-type measurement is performed on the beam. What is the probability of finding the value $-\hbar/2$?
(c) If the measurement of Sx was done first, independent of its outcome the measurement of S_y is done next. What is the probability of finding the value $-\hbar/2$?

QM I – 4

Consider a particle of charge *e* and mass *m* in constant crossed **E** and **B** fields:

$$\mathbf{E} = (0, 0, E), \qquad \mathbf{B} = (0, B, 0), \qquad \mathbf{r} = (x, y, z)$$

(a) Write the Schrödinger equation, in a convenient gauge.
(b) Separate variables and reduce it to a one-dimensional problem.
(c) Calculate the expectation value of the velocity in the x-direction in any energy eigenstate sometimes called the drift velocity.

QM I – 5

A particle of mass *m* and charge *q* sits in a harmonic oscillator potential $V = k(x^2 + y^2 + z^2)/2$. At time t=-∞ the oscillator is in its ground state. It is then perturbed by a spatially uniform time-dependent field

$$\mathbf{E}(t) = Ae^{-(t/\tau)^2}\hat{\mathbf{z}}$$

Where A and τ are constant. Calculate in lowest-order perturbation theory the probability that the oscillator is in an excited state at t=+∞.

SM – 1

Consider a molecule as a rigid motor with moment of inertia *I*. Its energy levels associated with rotation are given by

$$\varepsilon_j = \frac{\hbar^2}{2I} j(j+1)$$ with generacy $g_j = 2j+1$ and $j=0,1,2,...$

Show that the heat capacity per molecule associated with rotation is given by

$$C = 3k\left(\frac{2\theta_r}{T}\right)^2 \exp\left(-\frac{2\theta_r}{T}\right); \theta_r \equiv \frac{\hbar^2}{2Ik}$$

when $T<<\theta_r$. Reminder: $U = -\partial \ln Q / \partial\beta$.

SM – 2

Consider an ideal gas in a one-dimensional channel of length L. The energy of the particle is given by $E = \frac{p^2}{2m} - \varepsilon_o$.

(a) Show, using the classical approach, that the partition function of one particle is given by $Q_1(T,L) = \frac{L}{\lambda} e^{\varepsilon_o/kT}$. Reminder: $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2.$

(b) What is the partition function of N indistinguishable particles (just write down the answer)?

(c) Calculate the chemical potential of this system of N particles at temperature T.

SM – 3

Consider a system of N non-interacting particles that have two possible energy states, $E = 0$ or $E = \varepsilon$. Find the temperature of the system as a function of the total energy. What happens to the temperature of the system when the total energy is greater than $N\varepsilon/2$? Assume N to be a large number.

SM – 4

A gas obeys the following equation of state (the *Dieterici* equation):

$$P(v-b) = k_B T \exp\left(-\frac{a}{k_B T v}\right)$$

where $v=V/N$ and a and b are constants. Find the critical point (P_c, T_c, v_c) for this gas, if it exists.

SM – 5

A wire of length l and mass per unit length μ is fixed at both ends and tightened to a tension τ. What is the rms fluctuation, in classical statistical mechanics, of the midpoint of the wire when it is in equilibrium with a heat bath at temperature T? A useful series is

$$\sum_{m=0}^{\infty} (2m+1)^{-2} = \frac{\pi^2}{8}$$

QM II – 1

Consider a charged particle with charge q in a 2-D isotropic harmonic potential $V(x) = \frac{1}{2}Mw^2r^2$. A weak electric field E is applied along the diagonal direction (making 45 degree with x axis). (a) Using perturbation theory to calculate the ground state energy to the second order in E. (b) Solve the problem exactly and compare the result with part (a).

QM II – 2

Using the variation principle to estimate the ground state energy of the 1-D simple harmonic oscillator. Explain your choice of the trial wave function.

QM II – 3

Consider the scattering of a plan wave (with momentum k) by a 3-dimensional spherical potential.

(a) If the potential is a hard sphere with a radius R what is the phase shift and the total scattering cross section for the s-wave scattering.
(b) If the potential is such that the phase shift of s, p, d, wave scattering are □/2, □/4, □/6, what is the total scattering cross section.

QM II – 4

An isolated hydrogen atom has a hyperfine interaction between the proton and the electron spins ($\mathbf{S_1}$ and $\mathbf{S_2}$, respectively) of the form $J\,\mathbf{S_1}.\,\mathbf{S_2}$. The two spins have magnetic moments $\alpha\mathbf{S_1}$ and $\beta\mathbf{S_2}$, and the system is in a uniform magnetic field **B**. Consider only the orbital ground state.

(a) Find the exact energy eigenvalues of this system and sketch the hyperfine splitting spectrum as a function of magnetic field.

(b) Calculate the eigenstates associated with each level.

QM II – 5

A particle of total energy $E = h^2\alpha^2/(2m)$ moves in a series of N contiguous one-dimensional regions. The potential in the n^{th} region is $V_n = -(n^2-1)E$, where $n = 1,2,.....,N$

All regions are equal width □/〈 except for the first and the last, which are of effectively infinite extent. Calculate the transmission coefficients for a particle incident from either end.

EM II – 1

A thin, straight, conducting wire is centered on the origin, oriented along the z-axis and carries a current $\mathbf{I}=I_0\cos\omega_0 t\,\hat{z}$ everywhere along its length *l*. Define $\lambda_0 \equiv 2\pi c/\omega_0$.

a) What is the electric dipole moment of the wire?

b) What are the scalar and vector potentials everywhere outside the source region ($r \gg l$). State your gauge and make no assumptions about the size of λ_0.

c) Consider the potentials in the regime $r \gg l \gg \lambda_0$. Describe (qualitatively) the radiation pattern and compare it to the standard dipole case, where $r \gg \lambda_0 \gg l$.

EM II – 2

A ⊗$^+$ hadron decays at rest into a proton and a pion,

$$\Delta^+ \rightarrow p^+ + \pi^0 .$$

The rest mass of the ⊗ resonance is assumed to be $m_{⊗}$ = 1620 MeV/c^2, while the rest mass of the proton is m_p = 938 MeV/c^2 and the pion has m□ = 135 MeV/c^2.

(a) Using energy-momentum four-vectors, obtain the final state energy E_p, Lorentz factor ©$_p$, and the speed v_p/c for the proton.

(b) Obtain the comparable quantities for the pion, E□, ©□, and v□/c.

EM II – 3

Consider a circular current loop of radius *a* and of infinitesimal cross section that is confined to the z = 0 plane. Let there be a sinusoidally varying current $I\exp(-i\omega t)$ in the wire, giving rise to a complex amplitude for the current density

$$\vec{J}(\vec{x}') = I\sin(\theta')\delta(\cos(\theta'))\frac{\delta(r'-a)}{a}\vec{e}_\phi .$$

a) Show that the complex amplitude of the magnetic moment is

$$\vec{m} = \frac{1}{2c}\int d^3x'\,\vec{x}\times\vec{J}(\vec{x}') = \frac{\pi a^2}{c} I\vec{e}_z .$$

In the multipole expansion, a time-varying magnetic dipole gives rise to a vector potential field in the radiation zone of,

$$\vec{A} = ik\frac{e^{ikr}}{r}\left(\vec{n}\times\vec{m}\right) .$$

b) Use this to compute the distant ($kr \gg 1$) magnetic field and electric field.

c) Compute the angular distribution of the radiated power $dP/d\Omega$ and sketch the antenna pattern.

EM II – 4

The general expression for the radiated energy spectral-angular distribution of a relativistic electron is

$$\frac{dW}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int dt'\,\vec{n}\times(\vec{n}\times\vec{\beta})\exp[i\omega(t'-\vec{n}\cdot\vec{x}(t')/c)]\right|^2,$$

which is derived using the Lienard-Wiechart expression for the radiative part of the electric field.

Consider a nucleus that suddenly emits a beta particle. The sudden appearance of the beta decay electron is associated with a burst of electromagnetic radiation also, called appearance radiation. It arises because the electron's velocity and position are defined only for $t' > 0$:

$$\vec{\beta} = (0,0,\beta) \text{ for } t' > 0,$$

and

$$\vec{x} = (0,0,c\beta t') \text{ for } t' > 0.$$

With the observation direction taken to be $\vec{n} = (\sin\theta, 0, \cos\theta)$, show that the appearance radiation for beta decay is given by

$$\frac{dW}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\frac{\beta^2\sin^2\theta}{(1-\beta\cos\theta)^2}.$$

EM II – 5

A *tenuous plasma* consists of free electric charges of mass m and charge e. There are n charges per unit volume. Assume that the density is uniform and that the interactions between the charges may be neglected. Electromagnetic plane waves (frequency ω, wave number k) are incident on the plasma.

(a) Find the conductivity σ as a function of ω.
(b) Find the dispersion relation, i.e., the relation between k and ω.
(c) Find the index of refraction as a function of ω. The plasma frequency is defined by $w_p^2 \equiv 4\pi ne^2/m$. What happens if $\omega < \omega_p$?

Astro I – 1. Energy transport

The equation of radiative transfer in a plane parallel, gray atmosphere can be written as:

$$\cos\theta\frac{dI}{d\tau_\nu} = I - S$$

where I is intensity, τ_ν is the optical depth measured vertically from the surface, and S is the source function.

a) The source function S describes how propagating photons are removed and replaced by photons from the gas. Mathematically it is the ratio of the emission coefficient to the absorption coefficient. In local thermodynamic equilibrium it is equal to the Planck function B. Under what conditions (i.e. at what place in a star) is the intensity I also equal to B? Explain your answer.

b) Starting with the equation above, derive the equation of transport used in stellar interiors:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$

c) Use the condition for convection to show the limiting case for radiative transport is:

$$\frac{dT}{dr} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$$

(Hint: You will need the adiabatic relation $PV^{\lambda} = C$, and the equation of hydrostatic equilibrium to get the result in this form)

Astro I – 2. Stellar dimensional analysis

a) Use the equation of hydrostatic equilibrium in difference form to derive the dependence of stellar central pressure on total stellar mass and radius. Assuming an ideal gas equation of state, what is the mass and radius dependence of central temperature? (assume constant composition, homologous density profiles)

b) Now assume that nuclear fusion is a "perfect thermostat" that keeps the core temperature identical for all hydrogen burning stars. What is the predicted mass-radius relationship for the main sequence?

c) Use the equation of radiative transport in difference form to derive the mass-luminosity relationship under these assumptions. (You may use the approximation that $T_{central} - T_{surface} = T_{central}$)

d) Use the relations from b and c to predict the slope of the main sequence for "constant central temperature" stars (the observed value for real stars is between 7 and 8). Comment on this result and upon the importance of understanding nuclear burning to predict the slope of the main sequence.

Astro I – 3. Virial theorem

The virial theorem can be written as:

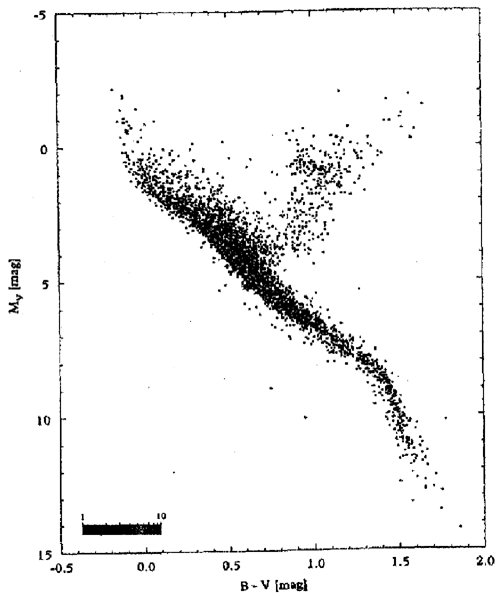
$$3\int PdV + \Omega = 0,$$

where P is the pressure and Ω is the gravitational potential energy. For an ideal, nonrelativistic gas this becomes $2K + \Omega = 0$.

a) Use the virial theorem to explain why adding significant energy to a star will cause it to cool.

b) We know a white dwarf will heat up if energy is added. How can this be consistent with the virial theorem? (Hint: the first term now has two components, one for the electrons and one for the ions)

Astro I – 4. Observational astronomy



a) The absolute v magnitude of the sun is about 4.8. Based on the Hipparcos H-R diagram (above), what is the B-V color of the sun?

b) Explain how to convert this B-V color into a temperature under the assumption that the sun is a blackbody.

c) What is the magnitude of a star with a B-V of 0.0? How many times more luminous than the sun is such a star?

d) If the sun is 6000K and has its spectral peak at 5500 angstroms, what is the temperature of a star with B-V of 0.0?

Astro I – 5. Nuclear Reactions

Nuclear reaction rates are proportional to

$$r \propto \left(\frac{8}{\mu\pi}\right)^{\frac{1}{2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) e^{\frac{E}{kT}\frac{b}{\sqrt{E}}} dE ,$$

where the *b* parameter is proportional to the product of the nuclear charges of the reactants and the square root of the reduced atomic mass $A=A_1A_2/(A_1+A_2)$

a) For non-resonant reaction rates, the S(E) can be treated as a constant, S_0, and the exponential approximated as a Gaussian. Explain where the two terms $e^{-E/kT}$ and $e^{-b/\mathrm{sqrt}(E)}$ come from. Sketch them separately and then sketch their product.

b) Show that the integrand has a maximum at $E_0 = (bkT/2)^{2/3}$

c) Helium burning is a two stage reaction, he first step of which is He^4+He^4=Be^8. It occurs at core temperatures about 10 times higher than for hydrogen fusion. How much higher in energy is the reaction peak? How is the peak otherwise changed?

d) The temperature dependence of the triple alpha reaction under discussion is T^{41}, much higher than the hydrogen burning sensitivity to T. Given your answer to c, how could this be true?

Numerical Constants:

Solar Mass (Msun):	1.989×10^{33} g
Solar Radius (Rsun):	6.96×10^{10} cm
Solar Luminosity:	3.847×10^{33} erg/s
Gravitational Constant (G)	6.6726×10^{-8} cm^3/g/s^2
Proton mass	1.6726×10^{-24} g = 938.27 MeV/c^2
Yield of p-p reactions (Q)	26.7 MeV = 4.28×10^{-5} ergs
Boltzmann constant	1.38×10^{-16} erg/K
Planck's Constant	6.626×10^{-27} erg-s
Electron mass	9.109×10^{-28} g
Proton mass	1.6726×10^{-24} g

NOTE: Due to the recent change of the graduate-level astronomy curriculum, the Astro II section of the 2007 Qualifier, which corresponded to the High-Energy Astrophysics class, will be replaced by a Galactic Dynamics section starting in 2008.

Astro II – 1. Wigner-Seitz Approximation

Consider a degenerate electron gas about an ion lattice.

(a) Consider a neutral, spherical cell of radius r_o about an ion of charge Z_e. Assume that the electrons are distributed uniformly and write down an expression for the charge q of the electrons within radius r.

(b) Calculate the potential energy E_{e-e} of the electron-electron interactions (i.e., the energy it takes to assemble a uniform sphere of Z electrons).

(c) Calculate the potential energy E_{e-i} of the electron-ion interactions.

(d) The total Coulomb energy of the cell is then $E_c = E_{e-e} + E_{e-i}$. Write down an expression for E_c as a function of Z and the electron density $n_e = 3Z/4\pi r_o^3$.

(e) The Coulomb correction to the ideal, degenerate electron gas pressure P_o is then $P_c = n_e^2 d(E_c/Z)/dn_e$. As electron density increases does $P/P_0 = (P_0 + P_c)/P_0$ increase, decrease, or stay the same (a) in the non-relativistic limit and (b) in the extreme relativistic limit?

Astro II – 2. White Dwarf Equilibrium and Stability

Consider a white dwarf of total energy $E=E_{int}+E_{grav}+E_{int}+E_{GR}$, where $E_{int} = AM \rho_c^{1/3}$ is the internal energy of an $n=3$ polytrope, $E_{grav} = -BM^{5/3}\rho_c^{1/3}$ is the Newtonial gravitational potential energy of an $n=3$ polytrope, $\Delta E_{int}=CM \rho_c^{-1/3}$ is the correction to the internal energy due to the electrons not being completely relativistic, $\Delta E_{GR}=-DM^{7/3}\rho_c^{2/3}$ is the correction to the gravitational potential energy due to general relativity, and M and ρ_c are the mass and central density, respectively. In cgs units, $A = 8.566*10^{14}(\mu_e/2)^{-4/3}$, $B = 4.264*10^{-8}$, $C = 4.950*10^{19}(\mu_e/2)^{-2/3}$, and $D = 4.549*10^{-36}$.

(a) Assume equilibrium and write down another relationship between A, B, C, D, M, and ρ_c.

(b) Ignore the correction terms in (a) and solve for M in solar masses. What is this mass?

(c) Do not ignore the correction terms and assume borderline instability to write down another relationship between $A, B, C, D, M,$ and ρ_c.

(d) Substitute (a) into (c) and eliminate $AM-BM^{5/3}$. Substitute (b) and eliminate M. Solve for ρ_c in g/cm^3.

(e) Inverse ß-decay occurs if $\rho_c \geq 1.14 * 10^9$ g/cm^3 for iron white dwarfs, $3.90 * 10^{10}$ g/cm^3 for carbon white dwarfs, and $1.37 * 10^{11}$ g/cm^3 for helium white dwarfs. Does inverse ß-decay or GR-induced instability terminate the sequence of (a) iron, (b) carbon, and (c) helium white dwarfs?

$M_{sun} = 1.99 * 10^{33}$

Astro II – 3. Pulsar Magnetic Dipole Model

Consider a neutron star that rotates at a frequency Ω with a magnetic dipole moment m that is oriented at an angle α to the rotation axis.

(a) The magnitude of m is $B_pR^3/2$, where B_p is the magnetic field strength at the magnetic pole and R is the radius of the neutron star. Write m as the sum of three orthogonal vectors, one along the rotation axis that depends on $|\mathbf{m}|$ and α , and two that also depend on Ω and time t.

(b) Calculate the rate at which the neutron star loses rotational energy: $\dot{E} = -2\,|\dot{\mathbf{m}}|^2 / 3c^2$.

(c) The neutron star's rotational energy is $E=I\Omega^2/2$, where I is the moment of inertia. Take a derivative and substitute into (b) to eliminate $\dot{E}$.

(d) Write an expression for the characteristic age of the pulsar: $T = -\Omega_0 / \dot{\Omega}_0$, where Ω_0 and $\dot{\Omega}_0$ are current values.

(e) Integrate (c) from Ω_i at $t=0$ to Ω_0 at $t=t_o$. Solve for t_o as a function of T, Ω_i, and Ω.

(f) For the Crab pulsar, T is measured to be 2556 years. Assume that $\Omega_i >> \Omega_0$ and calculate the pulsar's age. How accurate is your answer?

Astro II – 4. Neutron Star Accretion

Consider accretion onto a neutron star with a dipole magnetic field.

(a) The magnetic field will begin to dominate the flow of the in-falling gas at the Alfvén radius, where the energy density of the magnetic field becomes comparable to the kinetic energy density of the gas. Write down a simple expression for the energy density of the magnetic field in terms of field strength B and a simple expression for the kinetic energy density of the gas in terms of gas density ρ and speed v.

(b) For a dipole magnetic field, $B \approx \mu / r^3$, where μ is the magnetic moment. Assume that $v \approx v_{ff}$, the free-fall speed, and that $\rho \approx \dot{M} / 4\pi v_{ff} r^2$, where $\dot{M}$ is the accretion rate. Write down a simple expression for v_{ff} in terms of the mass M of the neutron star and r. Substitute these expressions into (a) and solve for the Alfvén radius $r=r_A$.

(c) As the in-falling gas flows to the surface, gravitational potential energy is converted to kinetic energy and when it strikes the surface the kinetic energy is converted to luminosity. Write down a simple expression for L in terms of M, $\dot{M}$, and R. Substitute this expression into (b) and eliminate $\dot{M}$.

(d) Take $\mu \sim 10^{30}$ cgs and L to be on the order of the Eddington luminosity. Ballpark r_A.

(e) For a dipole magnetic field, field lines are given by $\sin^2\theta / 2 = constant$. The in-falling gas is funneled to what fraction of the neutron star's surface?

$G = 6.67259 \times 10^{-8}\ cm^3\ g^{-1}\ s^{-2}$
$M_{sun} = 1.99 \times 10^{33}$

Astro II – 5. Aberration of Light

Consider the Lorentz transformation:

$$x'_{\parallel} = \gamma (x_{\parallel} - vt)$$
$$x'_{\perp} = x_{\perp}$$
$$t' = \gamma (t - v x_{\parallel} / c^2)$$

(a) Write down the velocity transformation.

(b) Let $\tan\theta' = u'_{\perp} / u'_{\parallel}$. Write down an expression for $\tan\theta'$ as a function of u, θ, and v (the aberration formula). Let $u=c$ and write down the aberration of light formula.

(c) HST images a star as it orbits at a speed of $v=7.56$ km/s, completing an orbit every 97 minutes. By how many arcseconds does the position of the star appear to change as the angle between the telescope's pointing and motion changes from -90° to +90°? Ground-based telescope have to track at a rate of 900 arcsec/min to compensate for the earth's rotation. At what average rate does HST have to "track" to compensate for aberration of light?

(d) Suppose that you are traveling through space at 1% of the speed of light. All objects with 90° of your direction of motion (half of the sky) will appear to be concentrated within how many degrees of your direction of motion? What if you are traveling at $\gamma = 100$?

Spring 2006 Qualifying Exam

(CM-1)
(a) What transformation is generated by type-two generating function $F_2^0 \equiv qP$?
(b) Show that the function

$$F_2 = F_2^0 + H(q, p[q, P])\Delta t$$

generates the motion in time for small intervals Δt. In other words, show that

$$Q(t) = q(t + \Delta t), \quad P(t) = p(t + \Delta t)$$

to first order in Δt.
(c) What "small" transformation on the six space and momentum components of a single particle is generated by the function

$$F_2 \equiv \vec{r} \cdot \vec{P} + \hat{n} \cdot \vec{L}\, \Delta\phi \quad ,$$

where $\vec{L} \equiv \vec{r} \times \vec{p}$ is the angular momentum? Show how you reach your conclusions.

(CM-2)
Show that if both the Hamiltonian H and a time-dependent quantity $G(q, p, t)$ are constants of the motion (e.g. $G = q - pt/m$ for a free particle) then $\frac{\partial G}{\partial t}$ is also a constant of the motion.

(CM-3)
Consider motion of a rod with its center-of-mass coordinates x and y and making and angle θ with the y axis. The Lagrangian is

$$L = 1/2m\dot{x}^2 + 1/2m\dot{y}^2 + 1/2I\dot{\theta}^2$$

and the rod is constrained to translate in the direction it is pointing by the nonholonomic condition

$$\dot{x}\cos(\theta) - \dot{y}\sin(\theta) = 0 \quad .$$

The initial conditions are $x(0) = y(0) = \theta(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = v_0$, $\dot{\theta}(0) = \omega_0$.
(a) Use a Lagrange multiplier to write down the equations of motion
(b) Find x, y, and θ as functions of time.

(CM-4)
A charged particle of mass m and charge e moves in the presence of an electromagnetic field with scalar potential Φ and vector potential $\vec{A}$. The Lagrangian is given by

$$L = \frac{1}{2}m|\vec{v}|^2 + \frac{e}{c}\vec{v}\cdot\vec{A} - e\Phi.$$

Recall that the physical fields are given by

$$\vec{B} = \vec{\nabla}\times\vec{A}, \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}.$$

(a) Obtain the Hamiltonian H and derive Hamilton's equations of motion for this system.
(b) Assume there is a uniform electric field $\vec{E}$ and a uniform magnetic field $\vec{B}$, which are mutually perpendicular. To be specific, let the electric field point in the y direction and the magnetic field point in the z direction. Assume the charged particle is initially at rest. Derive and solve the equations of motion.

(CM-5)
A mass m is constrained to a horizontal surface and connected to a wall by two identical springs with spring constant k. In equilibrium the mass, springs, and wall form an equilateral triangle, with the equilibrium spring length being l and the distance between attachment points on the wall also l. Let distance along the wall be given by x and distance perpendicular to the wall be given by y.
(a) Consider small amplitude motion away from equilibrium but confined to the plane. Derive and solve the equations for small amplitude motion. Obtain the two eigenfrequencies, ω_1 and ω_2, and find the eigenvectors.

Spring 2006 Qualifying Exam

(QM-1)

Demonstrate explicitly that the uncertainly relationship holds true for a 1D Simple Harmonic Oscillator in the state $|n>$.

(QM-2)

Given the normalized wave function for a one-dimensional system:

$$\psi(x) = (2\pi\alpha^2)^{1/4} \exp[-(x-\beta)^2/(4\alpha^2) + i(\gamma x - \delta t)]$$

(a) Obtain the average values of the following quantities:
energy $< E >$, squared coordinate $< x^2 >$ and momentum $< p_x^2 >$, force $< F_x >= d < p_x > /dt$, and probability current $J(x)$ in terms of the real constants α, β, γ, δ, and quote the expression used to obtain the answers.
(b) Show that there exist non-trivial conditions so that this system conserves probability.
(Useful information: a Gaussian function, with unit variance and whose integral is unity, is $(2\pi)^{-1/2} \exp[-x^2/2]$.)

(QM-3)

Prove the Thomas-Reiche-Kuhn sum rule

$$\Sigma_n \frac{2m|x_{n0}|^2}{\hbar^2}(E_n - E_0) = 1.$$

where the sum is taken over the complete set of eigenstates Ψ_n of energy E_n of a particle of mass m, which moves in a potential; Ψ_0 represents a bound state.

(QM-4)

Consider an ensemble of Hydrogen atoms in different states $|n, l, m, m_s>$. Restrict the ket space to $n = 1, 2$ only.
(a) The first measurement shows that 25% of atoms are in the state with

$n = 1$, while 25% each are in the states with $l = 1$ and $m = 1, 0, -1$, respectively. Assuming that the spin part is random, write down the density operator in the basis $|n, l, m, m_s>$.
(b) The atoms pass through a region with a weak magnetic field which only affects the spin, and after that the z-component of the spin is measured. It is found that 75% of the time the measurement yields a value of $+\hbar/2$. Write down the density operator after the second measurement. Explain the logic behind your answer.
(c) A third measurement is done to find out the z component of the total angular momentum of the atoms. What are the possible values and the associated probabilities?

(QM-5) Consider an ensemble of spin $S = 0$ composite particles, each of which is composed of two spin $S = 1/2$ particles. Let $P(\mathbf{a}+; \mathbf{b}+)$ be the probability that in a random selection an observer A measures the spin of a particle 1 along the direction $\mathbf{a}$ and finds it to be $+\hbar/2$, while an observer B measures the s[in of a particle 2 along the direction $\mathbf{b}$ and finds it to be $+\hbar/2$, etc.
(a) State the Bell's inequality (as described in the Sakurai's book) and its implications.
(b) Calculate explicitly the probabilities and show that the Bell's inequality is violated.

Spring 2006 Qualifying exam

(EM-1)
Show that the trajectory of a charged particle in a magnetic field can be duplicated by that of a current-carrying wire held at rest under constant tension (provided by some fixtures outside of the field region). Deduce the current I required in a wire of tension T to match the trajectory of a proton of momentum P.

(EM-2)
A spherical insulator (of radius a and dielectric constant ϵ) has its top hemisphere coated with a surface charge of constant density. Find the electrostatic potential everywhere inside and outside the sphere.

(EM-3)
Three point charges (q, $-2q$, q) are located on the z-axis and surrounded by a grounded, conducting spherical shell, as shown below.
(a) Write down the potential for the 3 charges in the absence of the grounded sphere. Find the limiting form as $a \rightarrow 0$, but the product $qa^2 = Q$ remains finite.
(b) Now add the grounded sphere and find the potential everywhere inside the sphere. Again, find the limit as $a \rightarrow 0$.

(EM-4)

A variable capacitor is connected to a battery of **EMF** $\mathcal{E}$. The capacitor initially has a capacitance C_0 and charge q_0. The capacitance is caused to change with time so that the current I is constant. Calculate the power supplied by the battery, and compare it with the time rate-of change of the energy stored in the capacitor. Account for any difference.

(EM-5)

It is well known that a static, uniform, magnetic field can do no work on either stationary or moving charges.

(a) Prove from definition.

It is also well known that a current carrying wire segment (current I, straight length vector L) will, in general, feel a force due to a static, uniform, magnetic field (which **can** do work).

(b) What is the magnitude and direction of that force? Provide annotated sketch.

(c) Finally, resolve the conflict between a) and b), since a current carrying wire segment, to all intents and purposes, is only an overall neutral collection of charges, some stationary and some moving.

Spring 2006 Qualifying Exam

(SM-1)

Consider an adsorption site for N_2 molecules. The adsorption site can accommodate up to two N_2 molecules with adsorption energy $\epsilon_A = -0.05$ eV for one molecule and $\epsilon_A = -0.02$ eV for two molecules ($\epsilon_A = 0$ for no molecule). Place this adsorption site in diffusive and thermal contact with an N_2 gas at 300 K with fugacity $z = \exp(\mu/k_B T) = 0.1$. Calculate the average number of adsorbed N_2 molecules on this site.

(SM-2)

Consider a two-dimensional classical ideal gas on a surface of $L \times L$ size. The energy of the particle is given by

$$H(q,p) = \frac{p^2}{2m} - \epsilon_0$$

where $\epsilon_0 > 0$ is the adsorption energy.

(1) Find the grand partition function of this system $Q(z, L, T)$ given the fugacity $z = e^{\mu/k_B T}$, L, and T.

(2) Determine the average number of adsorbed particles $< N >$ given the chemical potential μ, L, and T.

Useful information:

$\int_0^\infty e^{-x^2} x^{2\nu-1} dx = \frac{1}{2}\Gamma(\frac{\nu+1}{2})$; $\Gamma(n+1) = n!$

(SM-3)

A surface has a temperature of $800^0 C$. Atoms of sodium which strike the surface are found to be 90% ionized when they evaporate. Atoms of chlorine are found to be ionized negatively by one part in 10^6 when they evaporate from the same surface. What is the electron affinity of chlorine? The ionization potential of Na is $\phi = 5.1V$.

(SM-4)

Consider a system of three-dimensional rotators (with two degrees of freedom and no translational motion) in thermal equilibrium according to Boltzmann statistics; take account of the quantization of energy. Calculate the free energy, entropy, energy and heat capacity (per rotator) in the case of high temperature, making use of Euler's approximation formula:

$$\Sigma_{J=0} f(J+\frac{1}{2}) = \int_0^\infty f(x)dx + \frac{1}{24}[f'(0) - f'(\infty)] + ...$$

(SM-5)

A very simplified model for a metal is "jellium" in which a collection o e lectrons of number density n is neutralized by a uniformly distributed positive background of the same density. In Hartree-Fock theory the one-electron states are plane waves and the electron energy is the kinetic contribution and an exchange interaction with electrons of the same spin only. In the ground state, the electrons occupy all wave vector states up to the Fermi-wave vector k_F. For the paramagnetic material, as the temperature vanishes, $n = k_F^3/(3\pi^2)$ and the Helmholtz free energy per unit volume is $F/\Omega = k_F^5/(5\pi^2) - k_F^4/(2\pi^3)$ [in units of length $= a_B$ and energy $= e^2/(2a_B)$ where a_B is the Bohr radius].

Calculate the liquid density n, at coexistence with its vapor (vacuum), and the corresponding chemical potential for zero temperature. Note that the vapor, as T approaches zero, is not a degenerate electron gas (appropriately neutralized) but rather a, similarly neutralized, classical ideal gas - where does this enter your previous result?

Spring 2006 Qualifying Exam, QM II

(QM-1)
Consider the one-dimensional Schroedinger equation with $V(x) = \frac{m}{2}\omega^2 x^2$ for $x > 0$ and $V(x) = +\infty$ for $x < 0$.
Find the energy eigenvalues.

(QM-2)
A one-dimensional harmonic oscillator of mass m and frequency ω, which resides in its ground state for $t = -\infty$, is acted upon by a time-dependent force: $F(t) = F_0 \exp(-t^2/\tau^2)$ (parallel to the spring).
Derive the probability of finding the system in its first excited state at $t = +\infty$.
Useful information: $< n|z|n+1 >= [\hbar(n+1)/(m\omega)]^{1/2}$ and $\int_{-\infty}^{\infty} e^{-x^2} dx = \pi^{1/2}$.

(QM-3)
Consider an infinite potential well ($V(x) = 0$ for $|x| < a$ and $V(x) = +\infty$ elsewhere).
(a) Imagine that the potential on the left half of the well is raised to $V_0 > 0$. Use perturbation theory to calculate the ground state energy to first non-zero order in V_0.
(b) Imagine that the left part of the potential is raised to V_0 for only a finite time $0 < t < T$. Calculate the probability that the particle will be in the first excited state at $t = T$.

(QM-4)
A particle of mass m in the plane wave state $|\mathbf{k} >$ gets scattered off of a finite range spherical potential $V(r) = e^2/r,\ 0 < r < R$ and $V(r) = 0, r > R$.
(a) Use the first order Born approximation to calculate the differential cross section as a function of the scattering angle.
(b) In the low energy limit, what is the total scattering cross section?

(QM-5)

Consider a neutral Helium atom.

(a) Write down the Hamiltonian for this two-electron system.

(b) Neglecting any interactions between the two electrons, spin-orbit coupling and relativistic effects, what are the values (in eV) of the three lowest energy levels? List all the degenerate states for these three levels.

(c) If the interactions are considered, draw schematically the energy levels' diagram of the lowest energy levels. Write down explicitly the associated total wave functions (including both spatial and spin parts).

Spring 2006 Qualifying Exam, EM II

(EM-1)
An electron (initially at rest) is released from infinity and "falls" toward a nucleus with charge Ze. Assume that the electron is nonrelativistic and that the radiative reaction force on the electron is negligible.
a) What is the angular distribution of the emitted radiation?
b) How is the emitted radiation polarized?
c) What is the radiated power as a function of the distance between the electron and nucleus?
d) What is the total energy radiated when the electron reaches a distance r from the nucleus?

(EM-2)
The ordinary expression for Ohm's law is

$$\vec{J}' = \sigma \vec{E}',$$

where the current $\vec{J}'$ and electric field $\vec{E}'$ are denoted with a prime to mean the values observed in the rest frame of the medium. In relativistic terms, this is the frame in which the material's four-velocity appears to be

$$U'^{\mu} = (c, 0, 0, 0).$$

(a) Derive the relativistic generalization of Ohm's law that is valid in all frames. Note that the three-current $\vec{J}$ is part of the covariant four-current J^{μ}, so the generalization will be a four-vector expression. Note also that if the charge density ρ' in the rest frame of the material is non-zero, then in a boosted frame it will appear to contribute a convective current term.

(EM-3)
The general expression for the energy spectral-angular distribution of radiation from a relativistic charged particle is

$$\frac{dW}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int dt' \vec{n} \times (\vec{n} \times \vec{\beta}) \exp[i\omega(t' - \vec{n} \cdot \vec{x}(t')/c]\right|^2,$$

which was derived using the Lienard-Wiechart expression for the radiative part of the electric field.
For an observation direction $\vec{n}$, let there be two real orthonormal vectors $\vec{e}^A$ and $\vec{e}^B$ lying in the plane to which $\vec{n}$ is orthogonal. These three vectors make up an orthonormal triad. Hence the identity can be decomposed as

$$\delta_{ij} = n_i n_j + e_i^A e_j^A + e_i^B e_j^B,$$

and one finds the projection operator orthogonal to $\vec{n}$ to be

$$\delta_{ij} - n_i n_j = e_i^A e_j^A + e_i^B e_j^B.$$

Assume that a relativistic electron undergoes a collision and makes an abrupt change in its velocity from $\vec{\beta}$ to $\vec{\beta}'$. For frequencies ω that are low compared to τ^{-1}, where τ is the duration of the acceleration in the collision, show that the spectrum of Bremsstralung radiation is

$$\frac{dW}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c} \sum_{m=A,B} \left| \vec{e}^m \cdot \left(\frac{\vec{\beta}'}{1 - \vec{n}\cdot\vec{\beta}'} - \frac{\vec{\beta}}{1 - \vec{n}\cdot\vec{\beta}} \right) \right|^2 .$$

(EM-4)

Consider a cold plasma (thus ignoring spatial dispersion effects) but include the effects of collisions between the electrons and the background ions. Let the ion-electron collision rate be ν_{ie}. We can model the effect of collisions as a drag term in the equation of motion of the electrons:

$$m_e \frac{d\vec{v}}{dt} = -e\vec{E} - \nu_{ie} m_e \vec{v}.$$

(Here we ignore the response of the ions to the applied field.)
(a) Use this equation of motion to work out the dielectric response of transverse electromagnetic waves in a cold, collisional plasma.

An electromagnetic plane wave of frequency ω is traveling in a region of vacuum ($z < 0$) and is incident normally upon a plane interface of a half-infinite region of cold, collisional plasma ($z > 0$). Assume that the collision rate is small compared to the plasma frequency so that $\eta = \sqrt{\nu_{ie}/(2\omega_p)} \ll 1$.

(b) Work out the value of the reflectivity coefficient R of the plasma to first order in η when the frequency of the incident wave ω equals the plasma frequency ω_p.

(EM-5)

Two photons of the same energy E approach each other at an angle θ ($\theta = \pi$ corresponds to a head-on collision). Assume that the energy is just sufficient for the photons to create an electron-positron pair:

$$\gamma + \gamma \to e^- + e^+.$$

(a) Set up the various energy-momentum four-vectors and derive the condition on the energy E as a function of θ for this reaction to be at threshold.

(b) Derive the speed β of the electron and positron.

Spring 2006 Qualifying Exam, AS II

(AstroII-1)

Consider an ideal, cold $n - p - e$ gas in equilibrium.

(a) Write down the inverse β-decay reaction and the relation between the chemical potentials. Set $\mu_\nu = 0$ since the neutrinos escape.

(b) Using the fact that $\mu_i = E_{F,i}$, write down a relation between m_e, m_p, m_n, x_e,x$_p$, and x_n.

Fall 2005 Qualifying Exam

(EM-1)

Assume the presence of a cold plasma with electron number density n_e that is threaded by a uniform magnetic field (directed along the z-axis) of strength B_0. Circularly polarized plane waves with wave vectors $\vec{k}$ directed along the z-axis are incident on the plasma. Ignore particle collisions and ignore the effects of ions (i.e., consider only the motion of the electrons).
(a) Work out the dielectric response of the plasma to the plane waves and show that both circularly polarized modes are, in fact, eigensolutions in the magnetized plasma.
(b) Compute the phase velocities of the two modes.

(EM-2)

One means of creating pions is by colliding a gamma ray and a proton

$$\gamma + p \rightarrow n + \pi^+.$$

Assume that the gamma ray is directed along the x-axis and that the proton moves along the y-axis. In the lab frame, let the energy of the gamma ray be E_γ and assume that the proton has Lorentz factor γ_0.
In terms of the rest masses, m_π, m_p, and m_n, and Lorentz factor of the proton, γ_0, determine the condition on the gamma-ray energy E_γ such that the reaction is just at threshold for creating a pion and a neutron.

(EM-3)

A single harmonically oscillating dipole, with dipole moment $\vec{d}(t) = \vec{d}_0 e^{-i\omega t}$, is centered on the origin. This oscillating moment gives rise to a complex vector potential amplitude $\vec{A}(\vec{x}) = -ikr^{-1}e^{ikr}\vec{d}_0$ and a complex electric field amplitude

$$\vec{E}(\vec{x})_{\text{dipole}} = -k^2 \frac{e^{ikr}}{r} \vec{n} \times (\vec{n} \times \vec{d}_0)$$

in the radiation zone.
Assume the existence of two such dipoles, both oscillating at the same frequency. Assume that one dipole is displaced from the origin to $\vec{x} = \vec{x}_0$ and

the other is centered at $\vec{x} = -\vec{x}_0$. Let the first dipole have dipole moment $\vec{d}_0$ and the second have dipole moment $-\vec{d}_0$ (i.e., 180 degrees out of phase). Assume that $k|\vec{x}_0| \ll 1$.
(a) Obtain an expression for the electric field in the radiation zone of the combined source accurate to lowest order in k.
(b) Assuming that all of the components of $\vec{d}_0$ are in phase and that both $\vec{d}_0$ and $\vec{x}_0$ point in the z-direction, compute the angular distribution of radiated power, $dP/d\Omega$.
(c) What radiation multipole is this?

(EM-4)
A fast, charged particle with constant speed $v = c$ traverses a transparent, lossless glass sphere (index of refraction n) in vacuum. The particle moves along a diameter of the sphere, emitting Cerenkov radiation as it goes. What fraction of the total Cerenkov light is trapped by total internal reflection?

(EM-5)
A plane, lineraly polarized, electromagnetic wave propagates in the $+\hat{z}$ direction. A particle of charge q, mass m, and velocity $\vec{u}$ is acted upon by the wave.
(a) In terms of q, $\vec{u}$, c, and the electric field amplitude vector $\vec{E}_0$, what is the force $\vec{F}$ on the charge (at all z and t)?
(b) Discuss if it is possible that $\vec{F} = 0$ non-trivially (i.e., when q or $\vec{E}_0 \neq 0$) in terms of the vector $\vec{u}$.

Fall 2005 Qualifying Exam

(QM-1)

The Fermi-contact interaction, for hydrogenic atoms, is responsible for the ground state splitting of the singlet to triplet states. The perturbation Hamiltonian is

$$\Delta H = \frac{8}{3} g_+ g_- \beta_+ \beta_- (\vec{S}_+ \vec{S}_-) \delta(\vec{r}),$$

where g_+ and g_- are the spin g-factors, the s are the magnitudes of the magnetons, $\vec{S}_+$ and $\vec{S}_-$ are the spin vector operators, and $\vec{r}$ is the relative coordinate. For positronium (both $g_\pm = 2$), calculate the ground state singlet and triplet energy shifts, due to such a contact perturbation; you should write down the ground state wavefunctions from your knowledge of hydrogenic atoms (do not derive). Give the results in terms of $e^2/a_0 = 27.2eV$ and $e^2/(\hbar c) = 1/137$. (Use $\int_0^\infty x^n e^{-x} dx = n!$).

(QM-2)

Consider two identical particles with mass m and spin $1/2$ in a two-dimensional isotropic potential well: $V(x, y) = 0$ for $|x|, |y| < L$ and $V(x) = \infty$ elsewhere.
(a) If there is only one particle, what is the energy, degeneracy (consider both, spatial and spin), and parity symmetry of the ground and first excited states?
(b) If there are two particles, what is the energy, degeneracy (consider both, spatial and spin), and parity symmetry of the ground and first excited states?

(QM-3):

Consider scattering of a plane wave with momentum $\mathbf{k}$ off of a three-dimensional spherical δ-function potential $V(r) = (\gamma\hbar^2/2m)\delta(r - R)$.
(a) By solving the Schroedinger equation explicitly, show that the s-wave scattering phase shift is determined by the equation
$\tan(kR + \delta) = \frac{k \sin kR}{k \cos kR + \gamma \sin kR}$.
(b) Show that in the limit of low energies and at small phase sifts the s-wave scattering cross section is $4\pi\gamma^2 R^4$.

(QM-4)

Consider a three-level system described by the Hamiltonian

$$H = E_1(|1\rangle\langle 1| + |2\rangle\langle 2|) + E_3|3\rangle\langle 3| + \alpha(|1\rangle\langle 2| + |2\rangle\langle 1|) + \beta(|2\rangle\langle 3| + |3\rangle\langle 2|)$$

where $E_1 < E_3$. Theat the off-diagonal terms as perturbation and, using perturbation theory, obtain the ground state energy to second order in α and β.

(QM-5)

Calculate (semiclassically) the density of states $\nu(E) = dN(E)/dE$, where $N(E)$ is the number of energy levels with energies below E, for a one-dimensional potential $V(x) = V_0|x/a|^\eta$ (V_0 and η are positive) in the limit of large E.

Compare your approximate result with the exact ones available for $\eta = 2$ amd $\eta = \infty$.

Spring 2005 Qualifying Exam

(CM-1):

Consider a point particle orbiting a non-rotating black hole. This is a central force problem. The constants of motion number the same as the Newtonian Kepler problem. Using the first integrals for energy and angular momentum, a relativist gives you the following equation for radial motion

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right);$$

involving the "effective potential"

$$V_{\text{eff}}(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right).$$

Here E and L are constants of the motion (specific energy and specific angular momentum respectively), M is the mass of the black hole (constant), and r and τ mare radial and time coordinates.

(a) From this somewhat odd effective potential problem, give the conditions for the particle to be in circular orbit about the black hole.

(b) Solve these conditions for circular orbits. Do so by expressing the values that E and L must have in order that the orbit be circular at radius r (i.e., give $E = E(r)$ and $L = L(r)$).

(c) Sketch the effective potential. Are the circular orbits guaranteed to be stable or not?

(CM-2):

A particle of mass m is constrained to move on the surface of a sphere of radius $R(t)$. The radius of the sphere is in general a function of time (i.e., $dR/dt = \dot{R} \neq 0$) and is to be assumed as given and not a dynamical degree of freedom. The motion on the constraint surface can be described using the polar coordinates θ and ϕ. Assume that the potential energy $V(\theta, \phi)$ vanishes.

(a) Write down the energy and construct the Lagrangian L for two-dimensional motion on the constraint surface. Find the Euler-Lagrange equations of motion.

(b) Given the functional dependence of L, what constants of the motion will exist?
(c) Construct the Hamiltonian H. Is H equal to the energy E? Is H conserved?
(d) For motion initially confined to the equator ($\theta = \pi/2$, $\dot{\theta} = 0$), calculate how the velocity $R\dot{\phi}$ behaves in the presence of expansion.

(CM-3):

Spheres of radius r are projected at another (infnitely heavy) sphere of radius $R > r$ and scatter elastically.
(a) Find the dependence of the scattering angle θ on the impact parameter b.
(b) Calculate the differential cross section $\sigma(\theta)$.
(c) What is the total cross section?

(CM-4):

A particle moves in the plane according to the Lagrangian

$$L = m\dot{x}\dot{y} - a^2 xy \ ,$$

where a is a constant.
(a) Write down the equations of motion. What physical system do they represent?
(b) Show that L is invariant under the transformations

$$x' = e^{\epsilon} x \ , \quad y' = e^{-\epsilon} y.$$

(c) Use Noether's theorem to find the corresponding conserved quantity Q. (Recall that Noether's theorem deals with small ϵ.) What is the physical meaning of Q?

(CM-5):

Consider a transformation of coordinates in phase space:

$$Q = \sin q, \quad P = \frac{p - a}{\cos q}$$

where a is a constant.
(a) Use Poisson brackets to show that the transformation is canonical.
(b) Find a generating function $F_2(q, P)$ for the transformation.
(c) Suppose the Hamiltonian is $H = 1/2(q^2 + p^2)$ (an oscillator) and take the constant a above to be zero. Find the new Hamiltonian $K(Q, P)$ and write down Hamilton's equations in the new variables.

Spring 2005 Qualifying Exam

(SM-1):
Derive the following Maxwell equations:

$$(\frac{\partial T}{\partial V})_S = -(\frac{\partial P}{\partial S})_V, (\frac{\partial T}{\partial P})_S = (\frac{\partial V}{\partial S})_P, (\frac{\partial P}{\partial T})_V = (\frac{\partial S}{\partial V})_T$$

and

$$(\frac{\partial V}{\partial T})_P = -(\frac{\partial S}{\partial P})_T; \; TdS = C_V dT + T(\frac{\partial P}{\partial T})_V dV$$

and

$$(\frac{\partial U}{\partial V})_T = T(\frac{\partial P}{\partial T})_V - P$$

(SM-2):
Consider a gas contained in volume V, at temperature T. The gas is composed of N distinguishable particles of zero rest mass, so that energy E and momentum p of the particle are related by $E = pc$. The number of single-particle energy states in the range p to $(p + dp)$ is $\frac{4\pi V p^2 dp}{h^3}$. Find the equation of state and the internal energy of the gas and compare with an ordinary gas.

(SM-3):
10^{10} weakly interacting spinless particles, each with the mass of electron, are identical in appearance but obey classical statistics. They are confined in a cubical box which is 10^{-6} cm on the edge. Each particle undergoes a potential interaction with the box which is of two different sorts. One is attractive and leads to a bound state with the energy $-1eV$ which is well localized near the center of the box. The other interaction is a strong repulsion which prevents the particle from escaping through the walls of the box. Find at what temperature the pressure in the box is $1atm$.

(SM-4):

A solid contains N mutually noninteracting nuclei of spin 1. Each nucleus can therefore be in any of three quantum states labeled by the quantum number m, where $m = 0, \pm 1$. Because of electric interactions with internal fields in the solid, a nucleus in the state $m = \pm 1$ has the same energy $\epsilon > 0$, while its energy in the state $m = 0$ is zero.

Derive an expression for the entropy of the N nuclei as a function of the temperature T, and an expression for the heat capacity in the limit $\frac{\epsilon}{kT} \ll 1$.

(SM-5):

If a magnetic field H is applied to a gas of uncharged particles having spin $\frac{1}{2}$ and magnetic moment μ, and obeying Fermi-Dirac statistics, the lining up of the spins produces a magnetic moment/volume. Set up general expressions for the magnetic moment/volume at arbitrary T and H. Then for low enough temperatures, determine the magnetic susceptibility of the gas in the limit of zero magnetic field, correct to terms of order T^2. Note the integral:

$$\int_0^\infty \frac{\sqrt{E}dE}{\exp[(E-\xi)/kT]+1} = \frac{2}{3}\xi^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\xi}\right)^2 + ...\right]$$

Spring 2005 Qualifying Exam

(QM-1):

The dispersion of an observable A in a quantum state $|\alpha>$ is defined as $<(\Delta A)^2>=<(A-<A>)^2>$, where $<A>=<\alpha|A|\alpha>$ is the expectation value.

(a) Prove the Schwarz inequality $<\alpha|\alpha><\beta|\beta> \quad \geq \quad |<\alpha|\beta>|^2$;

(b) Show that the expectation value of a Hermitian operator is real and that of an anti-Hermitian operator is purely imaginary;

(c) Prove the uncertainty relationship $<(\Delta A)^2><(\Delta B)^2> \quad \geq \quad |<[A,B]>|^2/4$ for any pair of observables A and B.

(QM-2):

An electron moves in the presence of a uniform magnetic field in the z-direction ($\mathbf{B} = \mathbf{B}\hat{\mathbf{z}}$).

(a) Evaluate the commutator $[\Pi_x, \Pi_y]$, where $\Pi_i = p_i - eA_i/c$ and $\mathbf{A}$ is the vector potential associated with the magnetic field.

(b) By comparing the Hamiltonian with that of a harmonic oscillator obtain the exact energy eigenvalues.

(QM-3):

Consider an ensemble of non-interacting particles in a two-dimensional isotropic harmonic oscillator potential. The particles are known to be in the states associated with the lowest three energy levels with equal probabilities.

(a) How many distinct states are there?

(b) Write down the density operator in terms of Hamiltonian eigenkets.

(c) Calculate the ensemble average of $r^2 = x^2 + y^2$.

(QM-4):

Derive the dipole-dipole magnetic interaction energy of a proton and an antiproton at a fixed distance a, in eigenstates of total spin, in terms of the proton magnetic moment μ_0. Two magnetic dipoles have the interaction energy

$$V = \frac{1}{r^3}\left[(\underline{\mu}_1 \cdot \underline{\mu}_2) - 3\frac{(\underline{\mu}_1 \cdot \underline{r})(\underline{\mu}_2 \cdot \underline{r})}{r^2}\right]$$

(QM-5):

Use the variational principle to estimate the ground-state energy of a particle in the potential

V $= \infty$ for $x < 0$

V = cx for $x > 0$

Take xe^{-ax} as the trial function.

Spring 2005 Qualifying Exam

(EM-1):

1. a) Suppose that you found a magnetic monopole. How would you modify Maxwells equations (for free space) and the Lorentz force law to accommodate them? b) If the magnetic field is constant and uniform, show that the vector potential can be written as A = -1/2(r x B). What gauge have you chosen for this vector potential?

(EM-2):

A long straight wire carrying current I is placed a distance a above a semi-infinite magnetic medium of permeability μ. Calculate the force per unit length acting on the wire; be sure to specify the direction of the force.

(EM-3):

Calculate the capacity C of a spherical capacitor of inner radius R_1 and outer radius R_2, which is filled with a dielectric varying as

$$\epsilon = \epsilon_0 + \epsilon_1 \cos^2 \theta$$

where θ is the polar angle.

(EM-4):

Two parallel conducting plates are a distance 2b apart; one is at potential V, while the other is held at V. At z = 0, there is a grounded conducting plate, perpendicular to the other two and insulated from them. Calculate the electrostatic potential f(x,z) for the enclosed volume by (perhaps) working through the following steps:

a) Write down the potential f0(x) for large z.

b) Define f1(x,z) = f(x,z) - f0(x) and write down the differential equation for f1(x,z) and its boundary conditions.

c) Calculate f1(x,z) by using an appropriate series of orthogonal functions, and finally find f(x,z).

(EM-5):

Suppose that the electrostatic potential in empty space were governed by the equation

$$\nabla^2\phi - m\phi = 0$$

with m being a positive constant.

a) Find the solution of this equation in rectangular coordinates. b) Show that the solutions to this equation are unique. c) What is the appropriate solution for the situation shown below:

Winter 2004 Qualifying Exam

(EM-1):

Using the four-dimensional form of Green's theorem, solve the inhomogeneous wave equations

$$\Box^2 A_\mu = \frac{-4\pi}{c} J_\mu$$

(a) Show that for a localized charge-current distribution the 4-vector potential is

$$A_\mu(x) = \frac{1}{\pi c}\int \frac{J_\mu(\xi)}{R^2} d^4\xi$$

where $R^2 = (x-\xi)_\mu (x-\xi)^\mu$, $x_\mu = (x, y, z, it)$ and $d^4\xi = d\xi_1 d\xi_2 d\xi_3 d\xi_4$.
(b) From the definition of the field strengths $F_{\mu\nu}$ show that

$$F_{\mu\nu} = \frac{2}{\pi c}\int \frac{(J \times R)_{\mu\nu}}{R^4} d^4\xi$$

where $(J \times R)_{\mu\nu} = J_\mu R_\nu - J_\nu R_\mu$.

(EM-2):

A rectangular wave guide $(0 < x < a,\ 0 < y < b)$ has fields:

$$\mathbf{E} = -\hat{\mathbf{y}} H_0 \frac{\omega\mu a}{\pi} \sin(\pi x/a)\sin(kz - \omega t),$$

$$\mathbf{H} = H_0[\hat{\mathbf{x}}(ka/\pi)\sin(\pi x/a)\sin(kz-\omega t) + \hat{\mathbf{z}}\cos(\pi x/a)\cos(kz-\omega t)],$$

with: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ being unit vectors; x, y, z being coordinate distances, and $k^2 = \epsilon\mu\omega^2 - (\pi/a)^2$, i.e. a TE_{10} mode.
(a) Find the surface charge densities on the conducting walls.
(b) Find the surface current densities on the conducting walls.
(c) Find the time-averaged Poynting vector, $< S >$, and its space-average, $<< S >>$, across the wave guide cross-section.

(EM-3):

Consider an astrophysical region like a giant radio lobe associated with an active galaxy. In this volume of space there are relativistic electrons of Lorentz factor γ orbiting in a weak magnetic field. For simplicity, assume that the magnetic field $\vec{B}$ is uniform and that the orbits are planar and circular.
(a) Use the covariant Lorentz force equation to determine the position vector $\vec{x}(t)$ and the velocity of the $\vec{v}(t)$ of an electron. Using the covariant expression of Larmor's formula for the electric dipole power, or any other means, derive the expression for the total synchrotron power P_{synch}.
(b) How does the net synchrotron power depend upon the energy density u_B of the magnetic field?
Let the cosmic microwave background radiation have energy density u_{cmb} and take the average CMB photon energy to be $h\bar{\nu}$.
(c) Soft CMB photons will inverse Compton (IC) scatter from the relativistic electrons, creating high energy gamma rays and degrading the energy of the electrons. Give an order of magnitude estimate of the IC power P_{IC}.
(d) How does the inverse Compton power depend upon the energy density u_{cmb} of the microwave background?

(EM-4):

Consider two charged particles that either scatter off of each other or orbit each other in response to their electrostatic interaction. Let the first particle have charge q_1 and mass m_1 while the second particle has charge q_2 and mass m_2. Take the particles to have position vectors $\vec{r}_1$ and $\vec{r}_2$ with respect to their barycenter (center of mass).
(a) Find the electrostatic force on each particle (assume the motion is nonrelativistic) and set up the two vector equations of motion (differential equations).
(b) Obtain an expression for the net instantaneous radiated power in the dipole approximation valid at any point on the orbit. Hint: you need not solve the equations of motion.
(c) Show how the radiated power depends upon the charge-to-mass ratios of the two particles. Under what circumstances would two charge particles interact but emit no radiation in the dipole approximation? Explain.

(EM-5):

One of the dominant decay channels for the W vector boson is

$$W^+ \rightarrow e^+ + \nu_e,$$

which occurs 10.72 percent of the time. Assume that the W boson is at rest when it decays. Assume that the neutrino mass is zero.
(a) In terms of the W mass m_W and the electron mass m_e, give expressions for the energy $E_{(e)}$ and the Lorentz factor γ of the electron and the energy $E_{(\nu)}$ of the neutrino.
(b) Given the values $m_W = 80.42$ GeV (note GeV!) and $m_e = 0.511$ MeV, compute the values of $E_{(e)}$, γ and $E_{(\nu)}$.
(c) What is the difference $1 - \beta = 1 - v/c$ for the electron?

Winter 2004 Qualifying Exam

(QM-1):

Show that the partial wave decomposition of the scattering amplitude $f_{\mathbf{k}}(\hat{\mathbf{k}}')$ for scattering from initial momentum $\mathbf{k}$ to final momentum $\mathbf{k}'$

$$f_{\mathbf{k}}(\hat{\mathbf{k}}') = \frac{1}{k}\sum_{l=0}^{l=\infty}(2l+1)e^{i\delta_l(k)}\sin\delta_l(k)P_l(\hat{\mathbf{k}}.\hat{\mathbf{k}}')$$

is consistent with the optical theorem

$$\sigma = \frac{4\pi}{k}\mathrm{Im} f_{\mathbf{k}}(\hat{\mathbf{k}})$$

(QM-2):

Let two spin-half particles 1 and 2 have total spin $\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}$ and take as a basis for the 4-dimensional direct product spin space eigenvectors of S_{1z} and S_{2z}: $(\alpha_1\alpha_2,\ \alpha_1\beta_2,\ \beta_1\alpha_2,\ \beta_1\beta_2)$, where α, β have eigenvalues $S_z = +1/2, -1/2$ respectively.
Work out the 4×4 matrix $\mathbf{S}^2$ in this basis and show by explicit diagonalization that

$$\mathbf{S}^2(\alpha_1\beta_2 + \beta_1\alpha_2) = 2\hbar^2(\alpha_1\beta_2 + \beta_1\alpha_2), \quad \mathbf{S}^2(\alpha_1\beta_2 - \beta_1\alpha_2) = 0$$

(QM-3):

For a particle in a one-dimensional harmonic oscillator, $H = P^2/2m + m\omega^2x^2/2$, a coherent state $|\lambda>$ is defined as an eigenstate of the annihilation operator $\hat{a}$, $\hat{a}|\lambda> = \lambda|\lambda>$.
(a) Using the coherent state as a variational state, estimate the ground state energy of the particle in the potential $V(x) = (m\omega^2/2)(x^2 + \sqrt{2\hbar/(m\omega)}x)$.
(b) Calculate exactly the ground state energy in this potential.

(QM-4):

A hydrogen atom in its ground state $|nlm>=|100>$ is subjected to a time dependent (but spatially uniform) electric field, $E(t) = E_0 e^{-t/\tau}, \quad t > 0$, along the z-direction.
(a) Using the first order time dependent perturbation theory and symmetry considerations, show that there is only one non-zero transition probability between the ground state and the first four excited states: $|200>, |211>, |210>, |21-1>$.
(b) Calculate the non-zero transition probability (You do not need to evaluate any integrals explicitly.)

(QM-5):

For a time-independent Hamiltonian $\hat{H}$ and a trial wave function Ψ, we define the function

$$F(\lambda) = \frac{<\Psi|e^{-\lambda\hat{H}}\hat{H}|\Psi>}{<\Psi|e^{-\lambda\hat{H}}|\Psi>}$$

(a) Prove that $F(0) > E_0$ when $\hat{H}\psi_n = E_n\psi_n$ and $E_{n\neq 0} > E_0$ (this is, in fact, a well known result).
(b) Also, prove that $\lim_{\lambda\to\infty} F(\lambda) = E_0$
(one can, in fact, show that $dF/d\lambda$ is non-positive for all $\lambda > 0$).

Winter 2004 Qualifying Exam

(Astro-1):

Curve of growth:
A curve of growth is $\log(w_\lambda/\lambda)$ vs. $\log(n_x/\lambda)$, where n_x is the abundance of element x in a given stellear model atmosphere (defined by T_{eff}, $\log g$, and chemical composition.

1. Sketch a curve of growth qualitatively and quantify the three major regimes of $\log(w_\lambda/\lambda)$ vs. $\log(n_x/\lambda)$. Cite the approximate slopes of each regime.
2. Sketch the appearance of an absorption line profile as the abundances pass through each of the regimes. (Intensity from 0 to 1, with 1 as the continuum vs. λ.)
3.Once a line saturates at the line center, upon what does its residual intensity depend?
4. How does microturbulence affect a curve of growth? How can you use this behavior to estimate an appropriate value for the model atmosphere's microturbulence?
5. Compare the curves of growth for a dwarf star and a giant star, assuming identical T_{eff}, $\log g$, and chemical abundances. Explain why they differ.

(Astro-2):

Supernovae distances:
A type II supernova is discovered to be near maximum light in a distant galaxy. Its spectrum reveals essentially a blackbody spectrum with a number of superposed emission lines from a few elements, and in differing ionization states. At that time, $V = 22.0$ and $B - V = -0.10$.

1. How might the spectra be employed to estimate the surface temperature of the expanding remnant?
2. Suppose it is thereby determined that the temperature is $30,000K$. Assuming the colors can be well represented by the table of synthetic blackbody colors, what is the color excess ("reddening") for the supernova?
3. Model calculations predict that luminosity at this stage to be $L =$

$3 \times 10^9 L_\odot$. If the Sun's bolometric magnitude is +4.75, what is M_{bol} for the supernova?
4. Using the interstellar extinction law, what is the distance modulus, $(m - M)_0$, for the supernova?
5. Why does an infinite temperature blackbody have a finite $B - V$ color index?

(Astro-3):

Convection:
By analogy with a perfect, non-degenerate gas, Chandrasekhar defined adiabatic exponents for a gas with non-negligible radiation pressure as Γ_1, Γ_2, and Γ_3, where:

$$\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0$$
$$\frac{dP}{P} + \frac{\Gamma_2}{1 - \Gamma_2} \frac{dT}{T} = 0$$
$$\frac{dT}{T} + (\Gamma_3 - 1) \frac{dV}{V} = 0$$

for adiabatic changes of state.
1. Show that the condition for buoyant stability is:

$$\frac{1}{\Gamma_1} \frac{\rho}{P} \frac{dP}{dr} > \frac{d\rho}{dr}$$

(Hint: The density of an element displaced by dr, adiabatically, is $\rho^* = \rho(r) + \delta\rho_{adiabatic}$. Compare this to $\rho(r + dr)$, which is just $\rho(r) + (d\rho/dr)dr$).
2. Explain why this condition implies that the temperature gradient in the star must be less than the adiabatic gradient

$$|(\frac{dT}{dr})_{star}| < |(\frac{dT}{dr})_{ad}|$$

(Hint: think of the pressure of a displaced blob compared to $\frac{dP}{dr}$).
3. Show that this leads to the stability condition:

$$(\frac{dT}{dr})_{star} > (1 - \frac{1}{\Gamma_2}) \frac{T}{P} (\frac{dp}{dr})_{star}$$

(remember that $\frac{dT}{dr}$ and $\frac{dP}{dr}$ are both negative).
4. What happens to Γ_1, Γ_2, and Γ_3 in a region where gas is partially ionized? Are convections zones in a star more likely or less likely to be associated with partial ionization zones?

(Astro-4):

Equations of Stellar Structure:
Consider two stars with equal masses, but with radii that differ by a factor of two. Assume both are in hydrostatic equilibrium (or at least quasi-static equilibrium).
1. Derive the differential equation for hydrostatic equilibrium.
2. Using dimensional analysis, compare the expected central pressures of the two stars.
3. Assuming identical density profiles (with r) compare their central densities.
4. Assuming identical, non-degenerate core composition, compare their central temperatures.
5. Suppose the energy given off is provided by nuclear reactions at a rate of $E = E_0 \rho T^4$, compare their luminosities. Compare their expected surface temperatures.

(Astro-5):

Opacity/SAHA
1. Write the SAHA equation and explain what it has to do with the opacities in the Fig.1 appended below.
2. Explain why the opacities from each source (b-f, f-f, b-b, electron scattering) must be summed to get the effective opacities, but conductive opacities must be added thus: $\frac{1}{\kappa} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{cond}}$.

Winter 2004 Qualifying Exam

(HEAstro-1):

A neutron star of mass M and radius R is surrounded by a thin stationary accretion disk being fed mass at a rate $\dot{m}$. Consider a narrow annular section of the disk of inner radius r and radial width dr. As mass crosses from the outer radius to the inner radius of the annulus it gives up gravitational potential energy. Assume the disk is in equilibrium and all of the released energy is radiated from both the top and bottom surfaces of the disk.
(a) Give an expression for the local effective temperature of this annular section of the disk.
(b) Take the neutron star mass to be $M = 1.4$ $M_\odot$ ($M_\odot = 1.989 \times 10^{33}$ g), its radius to be $R = 10^6$ cm, and the inflow rate to be $\dot{m} = 10^{-9}$ $M_\odot$ yr^{-1}. Compute the value of the temperature of the disk near where the disk meets the surface of the star. (Ignore the additional energy release on the surface of the star.)
(c) Given this temperature, the mean photon energy has what value in electron volts? The emitted radiation lies in what band?

(HEAstro-2):

A particular high-mass X–ray binary (HMXB) is composed of an O type main sequence star of mass M_O in orbit with a neutron star of mass M_X. Accordingly $M_X \ll M_O$. The binary has an orbital period of P. Let the radius of the neutron star be R_X. Furthermore, assume that the O star emits a strong wind of total mass loss rate $\dot{m}_w$ and terminal velocity v_w.
(a) Give an expression (or expressions) that estimates the rate $\dot{m}_{\rm acc}$ at which mass is captured by and accreted onto the neutron star. What fraction of the mass lost by the O star ends up being accreted?
(b) Give an expression (or expressions) for the accretion luminosity L_X.

(HEAstro-3):

Consider a neutron star of mass M and radius R that is reasonably well modeled by an $n = 1$ Newtonian polytrope, which has a density profile given by

$$\rho(r) = \rho_c \frac{R}{\pi r} \sin \pi r/R,$$

where ρ_c is the central mass density. An $n = 1$ polytrope implies that

$$p(r) = K\rho(r)^2 = (p_c/\rho_c^2)\rho(r)^2,$$

where p_c is the central pressure. Ignore all general relativistic effects.
(a) Integrate the mass distribution to show that the central density is $\rho_c = \frac{\pi M}{4R^3}$.
(b) Integrate the equation of hydrostatic equilibrium to find the gravitational potential $\Phi(r)$ inside the star and the pressure profile $p(r)$.
(c) As part of solving (b), it is necessary to determine p_c and K. Express these parameters in terms of M, R, and Newton's constant G.
(d) Assume the neutron star has a mass $M = 1.4\ \mathrm{M}_\odot$ and radius $R = 11$ km ($M_\odot = 1.989 \times 10^{33}$ g). The inner crust of nuclei and superfluid neutrons gives way to the outer core of superfluid protons and neutrons at a density of 2.0×10^{14} g cm^{-2} (nuclear density). How far below the surface of the star is this transition layer?

(HEAstro-4):

Consider a massive CO white dwarf composed of equal amounts of carbon and oxygen by mass. Assume that the white dwarf has been slowly accreting mass from a companion star and now has a mass of $M = 1.25\ \mathrm{M}_\odot$ ($M_\odot = 1.989 \times 10^{33}$ g), a radius of $R = 1.8 \times 10^8$ cm, and has reached the point of instability. Assume moreover that just as collapse begins the central density reaches the point at which carbon burning is initiated. The binding energies per nucleon of carbon, oxygen and iron are 7.6 MeV, 8.0 MeV, and 8.8 MeV, respectively.
Given the above information, make a set of energy estimates to determine the fate of this star. Does it collapse to form a neutron star or does it explode? Justify your answer.

(HEAstro-5):

In the inner core of a neutron star, densities may be high enough to allow pions (π^-) to be in equilibrium with n, p, and e^-. In addition to beta equilibrium

$$p + e^- \rightarrow n + \nu_e, \quad n \rightarrow p + e^- + \bar{\nu}_e,$$

the reaction

$$n \rightarrow p + \pi^-$$

can occur but has a high threshold because of the pion rest energy of $m_\pi =$ 139.6 MeV. Pions are bosons so their chemical potential at zero temperature is $\mu_\pi = m_\pi c^2$, independent of density. (In fact, pions are speculated to form a Bose-Einstein condensate in neutron stars.) The rest energies of the proton, neutron, and electron are 938.3 MeV, 939.6 MeV, and 0.511 MeV, respectively.

(a) Write down the three equations that must be satisfied in nuclear equilibrium.

(b) Compute the values of the fermion relativistic factors x_e, x_p, and x_n at the threshold for π^- production. Give the ratio of neutron density to proton density at pion threshold.

Spring 2004 Qualifying Exam

(CM-1):

A particle of mass m moves in a plane under the action of a central force $f(r) = -dV(r)/dr$.
(a) Write down expressions for the angular momentum and energy in polar coordinates.
(b) Use conservation of angular momentum and energy to obtain an equation for the orbit $r(\theta)$.
(c) The orbit of the particle is a spiral of the form $r = r_0 e^{\theta}$. What is the dependence of the force $f(r)$ on r?

(CM-2):

Fluid flow and the Liouville theorem: Consider the flow of all points in Hamiltonian phase space corresponding to a system with N coordinates and N conjugate momenta. Each trajectory corresponds to a different set of intial conditions. Let $\rho(q, p, t)$ be the density of points, which can vary over phase space. [Here q and p are shorthand for all $2N$ q's and p's.] Because no trajectory can disappear, ρ must satisfy the continuity equation

$$\frac{\partial \rho(q,p,t)}{\partial t} + \vec{\nabla} \cdot [\rho(q,p,t)\ \vec{v}(q,p,t)] = 0,$$

where $\vec{v}$ is also a function of points on phase space and represents the velocity of a given point in phase space.
(a) Show that if the system obeys Hamilton's equation for some Hamiltonian H, the divergence of the velocity field $\vec{v}$ vanishes everywhere.
(b) Write down an expression for $\dot{\rho}(q, p, t)$, the **total** time derviative of the density around a point as that point moves along its trajectory. Use the continuity equation to show that the vanishing of the $\vec{\nabla} \cdot \vec{v}$ implies that $\dot{\rho} = 0$, i.e., that the density in a region doesn't change as the region moves.

(CM-3):

A bead of mass m slides down a parabolic wire that has a shape given by $y = \frac{1}{2}bx^2$.
(a) Write down the Lagrangian in the coordinates x and y.
(b) Obtain the equations of motion, incorporating the constraint relating x and y via a Lagrange multiplier λ. What is the physical meaning of λ?
(c) The bead is initially released from rest at a height y_0. Find the value of λ when the particle is at the origin. [**Hint:** This is easiest if you make direct use of energy conservation.]

(CM-4):

The Earth is in equilibrium between its internal pressure gradient and its own gravitational field. The mass and radius of the Earth are $M = 5.98 \times 10^{27}$ g and $R = 6.38 \times 10^8$ cm. Newton's gravitational constant is $G = 6.67 \times 10^{-8}$ dyne cm^2 g^{-2}. Assume that a seismic disturbance occurs at some point on the Earth's surface.
(a) Give order of magnitude expressions, in terms of M, R, and G, for the velocity v of seismic pressure waves and for the time T it takes for the pressure wave to pass through the Earth and reach the surface on the opposite side.
(b) Evaluate these expressions numerically and give order of magnitude estimates of v and T.

(CM-5):

Consider the transformation from variables q and p to a new set of variables Q and P given by

$$Q = Aq^n \cos p, \qquad P = Aq^n \sin p,$$

where A and n are constants.
(a) Use Poisson brackets to find what values of A and n are required in order for the transformation to be canonical.

Spring 2004 Qualifying Exam

(SM-1):

Show how a consideration of the appropriate partition functions leads to Fermi-Dirac and Bose-Einstein statistics. By analogy, find also the distribution function $f(\epsilon)$ resulting from "parastatistics", in which no more than **two** particles may occupy the same quantum state characterized by the energy eigenvalue ϵ.

(SM-2):

In a porous material an ideal gas of argon atoms (in the pores) is in equilibrium with argon atoms adsorbed on the internal surface (the surface of the pores). Per unit volume of the porous material, there are ρ_s sites at which the atoms can be adsorbed, and when the are, their energy is $-\epsilon$ per one adsorbed atom. Assuming that the particles behave classically, show that $\rho_{ad}/\rho_g = \rho_s \lambda^3 e^{\epsilon/k_B T}$. Here ρ_g is the gas density, ρ_{ad} is the number of atoms adsorbed per unit volume of the material, and $\lambda = \hbar/\sqrt{2\pi m k_B T}$ is the thermal wavelength of the argon atoms at temperature T.

(SM-3):

A harmonic one-dimensional oscillator has energy levels $E_n = \hbar\omega(n + 1/2)$ where ω is the characteristic frequency and the quantum number $n = 0, 1, \ldots$. Suppose that the oscillator is in thermal contact with a heat reservoir at temperature T such that $k_B T \ll \hbar\omega$.
(a) Find the ratio of the probability of the oscillator being in the first excited state to the probability of its being in the ground state.
(b) Assuming that only the ground and the first excited states are appreciably occupied, find the mean energy of the oscillator as a function of the temperature T.

(SM-4):

In plasmas, a degenerate electron fluid is neutralized by a uniform positive ion background of the same density. The Hartree-Fock internal energy of the system at $T = 0$ is (kinetic energy plus exchange):

$$U = \frac{e^2}{2a_0}\frac{(k_F a_0)^3}{3\pi^2}[\frac{3}{5}(k_F a_0)^2 - \frac{3}{2\pi}(k_F a_0)]$$

where $(k_F a_0)^3 = 3\pi^2 a_0^3 N/V$, N being the number of electrons, V the volume, and a_0 the Bohr radius.
Further, suppose that thermal effects can be neglected for the electrons while the positively charged ion background can be described as an ideal gas.
(a) What is the pressure due to the electrons?
(b) What is the total pressure (due to both the electrons and ions) as a fnction of temperature T?
(c) At $T \to 0$ the neutral electron-ion fluid can co-exist in two phases of different densities. What are these densities?

(SM-5):

An ideal gas has a temperature-independent molar specific heat C_v at constant volume. Let $\gamma = C_p/C_v$ denote the ratio of its specific heats. The gas is thermally insulated and allowed to expand quasi-statically from an initial volume V_i at temperature T_i to a finite volume V_f.
(a) Use the relation $PV^\gamma = const$ to find the final temperature T_f of the gas.
(b) Alternately, use the fact that entropy remains constant in this process to find the final temperature T_f.

Spring 2004 Qualifying Exam

(EM-1):

A conducting sheet at zero potential fills the $x - y$ plane. Sitting on the sheet is a hemispherical shell with inner radius a and outer radius b, centered on the origin (see Fig.1). The shell has a permanent electric polarization P, pointing in the z-direction. Find the electric potential for all $z > 0$.

(EM-2):

A ring of radius a has charge q uniformly spread on it. The ring is surrounded by a concentric sphere of radius b, held at zero potential. Find the potential for $a < r < b$.

(EM-3):

A quadrant of empty space ($x > 0, y > 0$ for all z) is bounded by two semi-infinite perfectly conducting planes that intersect along the line $x = y = 0$. The potential on the surfaces is held to $V = 0$. A charge q is located at the position $x = y = a, z = 0$.
Find the direction and magnitude of the force on the charge.

(EM-4):

(a) If a magnetic field is due entirely to a localized distribution of permanent magnetization, show that the integral taken over the entire space

$$\int d^3\vec{r}\vec{B}\vec{H} = 0$$

(b) The magnetic field between the pole pieces of a cyclotron is given by $B_z(r, z)$ where r is the distance from the central axis ($r^2 = x^2 + y^2$). If $|B_z|$ is a decreasing function of r, show that the field lines bow outward, as shown on Fig.2.

(EM-5):

A thin uniform metal disc lies on an infinite conducting plane. A uniform gravitational field is oriented normal to the plane. Initially the disc and plane are uncharged; charge is slowly added. What value of charge density is required to cause the disc to leave the plate?

Spring 2004 Qualifying Exam

(QM-1):

Answer following questions briefly:
(a) List two differences between quantum mechanics and classical mechanics.
(b) What are the differences between Schrodinger and the Heisenberg pictures.
(c) What is the significance of the Bell's inequality?
(d) What is the difference between a qubit and a bit? What makes quantum computing superfast?
(e) When two physical operator A and B commute, what does it imply? If they do not commute, what does it imply?

(QM-2):

A particle initially is in the first excited state of the infinite potential well: $V(x) = 0$, for $0 < x < L$, otherwise $V(x) = \infty$. At time $t = 0$, the left potential wall suddenly moves from $x = 0$ to $x = -L$.
(a) What is probability of finding the particle in the second excited state of the new potential well?
(b) If the energy is measured, what is the probability that its value is the same as that measured before the wall has moved?
(c) The answer found in b) is less than 1. Explain why this should be expected.

(QM-3):

Consider an ensemble of non-interacting spin 1/2 particles all of which has the orbital angular momentum $l = 2$.

(a) If the ensemble is random, write down the density operator matrix in the basis $|j, m>$.

(b) If all particles are in the identical quantum state $|j = 5/2, m = 3/2>$, write down the density operator matrix in the basis $|l, s; m_l, m_s>$.

(c) If the two ensembles in a) and b) have the same number of particles and are mixed together, and one measures the total angular momentum j, what is the probability that $j = 3/2$?

(QM-4):

A particle is in the ground state of a narrow one-dimensional potential well approximated as $U(x) = -\alpha\delta(x)$.

At $t = 0$ the potential well starts moving at a speed v. Find the probability that the particle remains in the ground state of the **moving** potential.

(QM-5):

Two particles with angular momenta $L_1 = 1$ and $L_2 = 2$ are in a state with a total anular momentum $L = 2$ and its z-component $L_z = 1$. Find the possible values of the linear combination $2L_{1z} - 3L_{2z}$ of the z-components of the particles' individual momenta and the probabilities of obtaining such values when measuring them in the above state.

Also, compute the average $< L = 2, L_z = 1|2\hat{L}_{1z} - 3\hat{L}_{2z}|L = 2, L_z = 1 >$.

Hint: make use of the following Clebsh-Gordon coefficients: $C^{L_1,L_2,L}_{m_1,m_2,m_1+m_2}$: $C^{1,2,2}_{-1,2,1} = 1/\sqrt{3}$, $C^{1,2,2}_{1,0,1} = -1/\sqrt{2}$, and $C^{1,2,2}_{0,1,1} = 1/\sqrt{6}$.

Spring 2003 Qualifying Exam

(CM-1):

A sphere of radius R is centered at the origin, which is also the center of a force field $\vec{F} = (\alpha/r^n)\hat{r}$, where α is a positive constant, $n \geq 3$, and $\hat{r}$ is the unit outward normal vector.
(a) What is the cross section $\sigma(E)$ for a projectile of energy E to hit the sphere? (It will help if you first make sure you understand the meaning of this cross section. There is only one sensible definition.)
(b) Is your answer negative for some energies? If so, what does the negative cross section signify?

(CM-2):

A ball is dropped from a vertical tower of height h in the northern hemisphere and lands a distance d_1 to the east of the tower's base. A second experiment is done in which another ball is launched from the ground at the base of the tower. The ball's initial velocity is strictly vertical and sufficient for it to reach a height equal to the tower before falling back down. What is the distance d_2 from the base where the ball lands and to which side of the tower does it land?

(CM-3):

A particle of mass m is constrained to move on the surface of a sphere, which is in a uniform gravitational field of strength g. The radius of the sphere is a given function of time $R(t)$ (i.e., $R(t)$ is not a dynamical degree of freedom).
(a) Choose coordinates and write down the Lagrangian.
(b) Find the canonical momenta and the Hamiltonian H.
(c) Is H equal to the energy E? Explain.
(d) Is H conserved? Explain. If $H \neq E$, is E conserved?

(CM-4):

A dynamical system consists of two degrees of freedom, x and y, in which (small) oscillations occur. The Lagrangian for the system is given by

$$L = L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\omega_0^2(x^2 + y^2) + \epsilon xy.$$

If the interaction term ϵxy were absent, the system would consist of two uncoupled oscillators that would both oscillate with frequency ω_0. The presence of the interaction term couples the oscillators.
(a) Find the eigenfrequencies, ω_1 and ω_2. Determine the time dependence of the normal modes of oscillation, $\xi_1(t)$ and $\xi_2(t)$.
(b) Using the relationship between the normal coordinates (ξ_1, ξ_2) and the original coordinates (x,y), obtain the time dependences of $x(t)$ and $y(t)$.

(CM-5):

A particle of mass m moves in two dimensions under the influence of a potential given in Cartesian coordinates by

$$V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2.$$

(a) Write down the Lagrangian for the system and find the conjugate momenta.
(b) Obtain the Hamiltonian for the system.
(c) Solve completely for the motion of the particle using the Hamilton-Jacobi approach. Determine the conserved quantities and their physical significance.

Spring 2003 Qualifying Exam

(SM-1):

The average energy of a system in thermal equilibrium is $< E >$.
(a) Prove that the mean square deviation of the energy is given by $< (E- < E >)^2 >= k_B T^2 C_v$, where C_v is the constant volume heat capacity of the system.
(b) Show that for a macroscopic system this deviation is negligible compared to the total energy of the system.

(SM-2):

Consider a group of 10^6 adsorption sites for O_2 molecules. Each adsorption site can accommodate at most one O_2 molecule with adsorption energy $\epsilon_A = -0.010eV$. Now, turn on a magnetic field $B = 100$ Tesla and place the adsorption sites in diffusive and thermal contact with an O_2 gas at 300 K with $\exp(\mu/k_B T) = 10^{-5}$ where μ is the chemical potential. The O_2 molecule has spin $S = 1$ with magnetic moment $\mu_B = 5.788 \times 10^{-5}$eV/Tesla).
Calculate the average number of sites which are occupied by O_2.

(SM-3):

How would the black-body radiation (BB) be changed if the photons obeyed the Fermi-Dirac (FD) statistics? Specifically:
(a) What is the would-be Planck radiation formula?
(b) How does the heat capacity C_v depend on temperature?
Next, consider our Universe pervaded by the $3K$ BB radiation. In a simple view, this radiation arose from the adiabatic expansion of a much hotter photon cloud which was produced during the Big Bang. Let us continue to assume that the photons obey the FD statistics.
(c) If in the next 10^{10} years the volume of the Universe increases by a factor of two, then what will be the temperature of the BB radiation?
(d) How much energy per unit volume is contained in this cloud of radiation?

Estimate the result within an order of magnitude in J/m^3.
Use: $k_B = 1.4^{-23} J/K$, $\hbar = 10^{-34} Js$, and $c = 3 \times 10^8 m/s$.

Derive (not simply guess!) your results (e.g., start by considering a harmonic oscillator with only two levels), but don't do any complicated dimensionless integrals.

(SM-4):

A cloud chamber contains water vapor at its equilibrium vapor pressure $P_\infty(T_0)$ corresponding to an absolute temperature T_0. Assume that
1) the water may be treated as an ideal gas; 2) the specific volume of water may be neglected compared to that of vapor; 3) the latent heat l of condensation and $\gamma = C_p/C_v$ may be taken constant; $l = 540 cal/g$, $\gamma = 3/2$.
(a) Calculate the equilibrium vapor pressure $P_\infty(T_0)$ as a function of the absolute temperature T.
(b) The water vaport is expanded until the temperature is T, $T < T_0$. Assume the vapor is now supersaturated. If a small number of droplets of water is formed, what is the equilibrium radius of these droplets?

(SM-5):

(a) Calculate the grand partition function $Z(z, V, T)$ for a two-dimensional ideal Bose gas and obtain the limit

$$\lim_{V \to \infty} \frac{\log Z(z, V, T)}{V}$$

where $V = L^2$ is the area available to the system.
(b) Find the average number of particles per unit area as a function of z and T.
(c) Show that there is no Bose-Einstein condensation for a two-dimensional ideal Bose gas.

Spring 2003 Qualifying Exam

(EM-1):

A line charge of length $2d$ with a total charge Q has a linear density varying as $(d^2 - z^2)$ where z is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b > d$ is centered at the midpoint of the line charge.
Find the potential $\Phi(\vec{r})$ everywhere inside the spherical shell as an expansion in terms of the Legendre polynomials.

(EM-2):

Consider a spherically symmetric potential due to a surface charge on a sphere of radius a

$$\Phi(r < a) = \frac{q}{4\pi\epsilon_0\sqrt{ar}}; \qquad \Phi(r > a) = \frac{q}{4\pi\epsilon_0 r}$$

(a) What is the total charge on the surface of the sphere of radius a?
(b) A point dipole in vacuum has the potential

$$\Phi(\vec{r}) = \frac{\hat{r}\vec{p}}{4\pi\epsilon_0 r^2}$$

If the dipole is aligned along the $+\hat{z}$ axis and enclosed is a cavity of radius R within an infinite grounded conductor,
1) find $\Phi_{tot}(\vec{r})$ for $r < R$ by solving the Laplace equation in spherical coordinates;
2) find $\sigma(\theta)$ on the cavity surface.

(EM-3):

(a) Write down the Maxwell's equations in vacuum, with sources, in a differential form.
(b) Show that in terms of the scalar potential Φ and the vector potential $\vec{A}$,

two of these equations are automatically satisfied.
(c) Describe the gauge invariance of the two remaining equations for Φ and $\vec{A}$.
(d) Show that even after imposing the Coulomb gauge these equations are still invariant under some residual gauge transformations.

(EM-4):

Laplace's equation in cylindrical coordinates is

$$\frac{\partial^2\Phi}{\partial\rho^2}+\frac{\partial\Phi}{\rho\partial\rho}+\frac{\partial^2\Phi}{\rho^2\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

where $r=\sqrt{\rho^2+z^2}$.
(a) Solve for $\Phi(\rho,\phi,z)$ by using a separation of variables. Find the three ordinary differential equations, solve them explicitly for the ϕ- and z-dependence, and discuss the natural solutions of the radial equation.
(b) Find $\Phi(\rho,\phi,z)$ inside the cylinder of radius a and height L that satisfies the following boundary conditions:
$\Phi(\rho,\phi,z=0)=0$ for $0\leq\rho\leq a$; $\Phi(\rho=0,\phi,z)=0$ for $0\leq z\leq L$; $\Phi(\rho,\phi,z=L)=V(\rho,\phi)$ for $0\leq\rho\leq a$.

(EM-5):

Compute the total magnetic field induced by an external field $\vec{B}_{ext}=B_0\hat{z}$ inside a uniformly magnetized sphere of radius R by introducing a "magnetic scalar potential" $\vec{H}=-\nabla W$ (this is possible because $\nabla\times\vec{H}=\vec{J}=0$) and using the separation of variables by analogy with the electrostatic Poisson equation.

Spring 2003 Qualifying Exam

(QM-1):

A coherent state of a one-dimensional simple harmonic oscillator is defined as the eigenstate of the annihilation operator $\hat{a}^-$. Prove that
$|\lambda >= \exp(-|\lambda|^2/2)\exp(\lambda a^+)|0>$
is a normalized coherent state with eigenvalue λ.

(QM-2):

A composite particle is made of two spin 1 particles.
(a) What are the possible values of the total spin of the composite?
(b) If the composite is in the state $|S = 2, S_z = 0>$ and one measures $\hat{S}_{1z}$, what are its possible values and the associated probabilities?

(QM-3):

A particle initially is in the ground state of the infinite potential well
$V(x) = 0$ for $0 < x < L$; otherwise $V(x) = \infty$.
At time $t = 0$, the right potential wall suddenly moves from $x = L$ to $x = 2L$.
(a) What is the probability of finding the particle in the ground and the first excited state of the new potential well?
(b) If one measures the position of the particle, what is its expectation value at $t = 0$? What is the expectation value at $t = \infty$?

(QM-4):

An electron spin is placed in a magnetic field $\vec{B} = B\hat{z}$. At $t = 0$ the spin is in the state with a definite value $+\hbar/2$ of the operator $\hat{S}_y$.
Find the (time-dependent) average value $< \hat{S}_x(t) >$ at $t > 0$ and the probabilities $P_\pm(t)$ of measuring S_x and obtaining $+\hbar/2$ and $-\hbar/2$, respectively.

(QM-5):

Derive the uncertainty relation

$$\sigma_x \sigma_H > \frac{\hbar}{2m} | < \hat{p} > |$$

where σ_x and σ_H are the variances of the coordinate operator $\hat{x}$ and the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(x)$.
What does this relation turn into in the case of a stationary state?

Winter 2003 Qualifying Exam

(QM-1):

Consider spin-1/2 electrons in the one-dimensional infinite potential well ($V(x) = 0$ if $|x| < L/2$ and $V(x) = \infty$ if $|x| > L/2$).
1. If there is only one electron, what are the ground state energy and its degeneracy? Under parity symmetry operation, is the ground state even or odd?
2. If there are two non-interacting electrons, what are the energy, degeneracy, and parity of the ground state?
3. If there are two electrons with the interaction $V = -J\mathbf{S_1} \cdot \mathbf{S_2}$ where $J > 0$ is much larger than the energy level spacing of the potential well, then what are the energy, degeneracy, and parity of the ground state?

(QM-2):

Consider a particle in an anisotropic two-dimensional harmonic potential $V(x, y) = \frac{1}{2}m(\omega_1^2 x^2 + \omega_2^2 y^2)$ where $\omega_1 < \omega_2 < 2\omega_1$.
A) What is the ground, 1st, 2nd excited state energy?
B)At time $t = 0$ the particle is in the ground state and the perturbation $V(t) = xe^{-t/\tau}$ is turn on. Using the time dependent perturbation theory, calculate the probability that the particle will be in the first excited state at time $t \to \infty$.

(QM-3):

Consider elastic scattering of a particle of mass m which is initially in a plane wave state with the wave vector k off of a spherically symmetric potential $V(r) = -V_0/r$ for $a < r < b$ and $V(r) = 0$ elsewhere. Using the first order Born approximation, calculate the differential scattering cross section as a function of the angle θ between the incoming and the outgoing plane waves.

(QM-4):

Let Ψ be the variational trial function for the ground state Ψ_0 of a system with non-degenerate energy eigenvalues. Assume that Ψ and Ψ_0 are real, normalized to unity, and that $\int \Psi\Psi_0 d\tau$ is positive.
Show that

$$\frac{1}{2}\int |\Psi - \Psi_0|^2 d\tau \leq 1 - (1 - \frac{< H > - E_0}{E_1 - E_0})^{1/2} \approx \frac{1}{2}\frac{< H > - E_0}{E_1 - E_0}$$

where E_0 and E_1 are the exact energies of the ground and first excited states, and $< H >$ is the expectation value of the Hamiltonian in the state Ψ.

(QM-5):

A particle is in the ground state of a one-dimensional δ-functional potential $U(x) = -\alpha\delta(x)$. At $t = 0$ the potential starts moving at a speed v.
Find the probability that the particle remains in the ground state of the moving potential.

Winter 2003 Qualifying Exam

(EM-1):

Show how the residual gauge symmetry for the potentials $A_\mu = (\vec{A}, i\Phi)$ which satisfy the Lorentz condition can be used to prove that in four space-time dimensions the photon has only two physical degrees of freedom.

(EM-2):

Show that it is possible for electromagnetic waves to be propagated in a hollow metal pipe of rectangular cross section with perfectly conducting walls. What are the phase and group velocities?
Show that there is a cutoff frequency below which no waves are propagated.

(EM-3):

Using conservation of energy and momentum for a system of charged particles and electromagnetic fields (but without invoking special relativity), derive the expression for the 3-vector which represents the energy flow of the electromagnetic field (Poynting vector). Then discuss how this 3-vector becomes part of a covariant 4-vector in a relativistic treatment.

(EM-4):

If neutrons from a cosmic ray interaction one light-year from Earth were to reach here with a probability $1/e$ or greater, what must their minimum energy be?
If they decay, what is the maximum angle to the flight path at which their decay electrons could be produced?

Hint: Neutron decays $n \to p + e^- + \overline{\nu}_e$. The rest lifetime is $\tau_n = 886$ seconds. Use the masses $M_n = 939.6MeV, M_p = 938.3MeV, M_e = 0.5MeV$, and $M_{\overline{\nu}} = 0$.

(EM-5):

Find time dependence $\mathcal{E}(t)$ of the energy of a relativistic charge of mass m rotating in a constant uniform magnetic field B due to electromagnetic radiation.

Hint: apply the so-called Lienard's generalization of the (non-relativistic) Larmor formula for the total radiation power in the case of the perpendicular $\vec{v}$ and $\vec{a}$.

Winter 2003 Qualifying Exam

(HEAstro-1):

The giant radio lobes associated with radio galaxies like Cyg A and Her A are observed because of synchrotron emission of relativistic electrons in a magnetic field. The radio flux, when combined with inferred distance of the galaxy, allows us to compute the total synchrotron luminosity L. The synchrotron emission depends upon the strength of the magnetic field B, the total number of relativistic electrons N_e, and their Lorentz factor γ. Assume that the total energy E_T in the radio lobe is the sum of the relativistic electron energy E_e and the magnetic energy E_B. The observed synchrotron flux from Her A, for example, implies a minimum total energy in the lobe of $E_T \simeq 10^{60}$ ergs.
Assume that $\gamma \gg 1$ and let the volume of the radio lobe be V.
1. Derive the expression for the total synchrotron luminosity L.
2. Express the luminosity in terms of the magnetic energy density and the relativistic electron energy density.
3. Obtain the condition for minimum total energy in the lobe based on fixed synchrotron luminosity and Lorentz factor γ.

(HEAstro-2):

Consider the degenerate electron equation of state in the relativistic limit (hint: $\phi(x) \simeq x^4/12\pi^2$). Assume that a star with this equation of state (nearing the Chandrasekhar mass) has $\mu_e = 2$.
1. Work out the effective polytropic relationship between pressure and mass density. That is, compute the K and γ that give the correct limiting relationship $p = K\rho^\gamma$.
2. Using dimensional analysis, express the pressure and mass density in terms of the star's mass M and radius R.
3. Using parts (1) and (2) above, derive an approximate expression for the Chandrasekhar mass M_c in terms of h, c, G, and m_p. Compute the numerical value of this approximation.

(HEAstro-3):

Using dimensional analysis (or self-similarity) find the Sedov-Taylor type solution for a *cylindrical* blast wave. Assume that a large amount of energy per unit length, $\sigma = E_T/L$, is injected instantaneously along an infinite line and that a shock wave expands into a quiescent medium of mass density ρ_0.
1.Determine how the radius R, velocity v, and pressure p of the cylindrical shock scale with time.
2. If the quiescent medium has a pressure p_0, determine the radius at which the self-similar solution (or simple dimensional analysis) breaks down.

(HEAstro-4):

Consider a stationary, thin accretion disk surrounding a compact star. You may assume that the disk thickness is proportional to radius, $h(r) = \alpha r$, with $\alpha \ll 1$. Likewise, you may assume that the inspiral of gas is gradual, with the magnitude of the radial velocity being a small fraction of the azimuthal (orbital) velocity: $v_r = \alpha v_\phi$. Assume that energy is radiated away locally from the surface of the disk just as fast as it is released. Assume local black body emission from each spot on the disk.
1. Derive the dependence between effective temperature T_{eff} and radius r (i.e., determine the temperature profile of the disk).

(HEAstro-5):

Say you have a radio lobe that is inflated with relativistic electrons that are emitting synchrotron radiation. Assume there is no active injection of fresh relativistic electrons and there is no additional acceleration occurring. The electrons are merely aging because of their radiative losses. Take the magnetic field to be of uniform strength B.
1. Using the expression for total synchrotron power, calculate the time dependence of the energy of electrons whose initial energy is $E_0 \gg m_e c^2$.
2. Assuming that the relativistic electrons have a spectrum $N_0(E)$ which initially extends to infinite energy, define a "cutoff" energy for these particles as a function of time.

3. For synchrotron radiation, the critical frequency $\nu_{\rm crit}$, where the spectrum of radiation emitted by an electron of energy E peaks, is given by

$$\nu_{\rm crit} = CBE^2,$$

where $C = 3e/4\pi m_e^3 c^5$ is a constant. Use this information to define a "cutoff" frequency of the radio lobe containing aged relativistic electrons.

Spring 2002 Qualifying Exam

(CM-1):

Using the Hamilton-Jacobi method, solve the problem of the motion of a point particle of mass m that is under the influence of a uniform vertical gravitational field and is confined to a vertical plane. Let x be the horizontal coordinate and y be the vertical coordinate. Assume that the particle is fired off at time $t = 0$ from the origin, $x = 0$ and $y = 0$, with an initial velocity $\vec{v}_0$ that makes an angle α with respect to the horizontal.

(1) Solve the Hamilton-Jacobi equation for the principal function S and characteristic function W. Obtain the constants of integration and evaluate them in terms of the initial values given above.

(2) Find both the trajectory of the projectile $y(x)$ and the time dependence of both of its Cartesian coordinates (i.e., give $x(t)$ and $y(t)$).

(CM-2):

Consider the motion of a particle of mass m, under the influence of gravity, that is constrained to move without friction on the surface of a paraboloid of revolution (i.e., a parabolic bowl). The paraboloid lies open upward with a shape given by

$$y = \frac{1}{2}ar^2,$$

where y is the vertical height and r is the cylindrical radial distance from the axis of symmetry. Let ϕ be the angular coordinate.

(1) Write down the Lagrangian describing motion on the surface and determine its symmetries. Compute the conjugate momenta.

(2) Obtain the energy equation for radial motion and the effective potential. Write down the equation that determines the turning points of radial motion. Show how many real solutions this equation has.

(3) Obtain the conditions for circular motion within the bowl. Derive the physical properties of the circular orbit (energy, angular momentum, period, etc) whose radius is some r_0.

(CM-3):

Consider the longitudinal oscillations of two identical masses (m) that are held between two walls by three identical springs (k). Working from first principles:
(1) Find the Lagrangian and Lagrange's equations.
(2) Obtain the eigenfrequencies and eigenvectors. Sketch the motions.
(3) Obtain the normal coordinates and diagonalize the Lagrangian.

(CM-4):
Let $\vec{L} = \vec{r} \times \vec{p}$ be the angular momentum vector. Its components are

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x.$$

(1) Compute all combinations of the Poisson brackets $[L_i, L_j]$. (2) Using the previous result, calculate the brackets $[L_i, L^2]$.

(CM-5):
Consider the transformation

$$Q = \sqrt{e^{-2q} - p^2}, \qquad P = \cos^{-1}(pe^q).$$

Determine whether or not this transformation is canonical.

Spring 2002 Qualifying Exam

(SM-1):
The average energy of a system in thermal equilibrium is $< E >$. Prove that the mean square deviation of the energy is given by the formula $< (E- < E >)^2 >= kT^2 C_v$ where C_v is the heat capacity of the entire system at constant volume. Use this result to show that the energy of a macroscopic system may ordinarily be considered constant when the system is in thermal equilibrium.

(SM-2):
Calculate the grand partition function for a system of N non-interacting quantum mechanical harmonic oscillators, all of which have the same natural frequency ω_0. Do this for the following two cases:
(a) Boltzmann statistics;
(b) Bose statistics.

(SM-3):
Consider a lattice of N atoms of spin 1/2 described by quantum mechanical operators $\vec{\sigma}_i$ which form a triplet of the Pauli matrices. The Heisenberg model of ferromagnet has the Hamiltonian

$$H_H = -\epsilon \sum_{<ij>} \vec{\sigma}_i \vec{\sigma}_j - \mu \sum_{i=1}^{N} \vec{\sigma}_i \vec{h}$$

where $< ij >$ denotes nearest-neighbor pairs, $\vec{h}$ is a uniform magnetic field, and $\epsilon, \mu > 0$. Another model, the Ising one, has the Hamiltonian

$$H_I = -\epsilon \sum_{<ij>} S_i S_j - \mu \sum_{i=1}^{N} S_i h_z$$

where $S_i = \pm 1$ are the Ising spins and h_z is the z-component of $\vec{h}$. Use the variational principle to prove that, for a given temperature T, the free energy of the Heisenberg model is not greater than that of the Ising model.

(SM-4):

Two vessels, A and B, contain an ideal gas of N and $2N$ molecules, respectively. Initially, the vessels are thermally isolated from each other and have temperatures T_A and T_B. The vessels are now brought into thermal contact. Find the change in entropy of the entire system after the new equilibrium is established.

(SM-5):

Consider a lattice with N_A (Avogadro number) atoms. The solid circles represent the normal positions of the atoms whereas the open circles represent the positions of interstitial defects. Both the numbers of normal and defect positions are N_A. Assume that is costs energy U to move an atom from a normal to a defect position. Find the number of occupied defect sites n (this is also a huge number ($n >> 1$) even when it is only a small fraction of N_A) at temperature T in the limit $n \ll N_A$. (Ignore all the other effects such as phonons, etc.).

Spring 2002 Qualifying Exam

(EM-1):

A charge q is situated a distance d away from the semispace ($z < 0$) filled with uniform linear dielectric material of susceptibility χ_e. Use the method of images to find the total charge induced on the surface of the dielectric medium and the force acting on the charge q.

(EM-2):

Find the electric displacement $\vec{D}$, the electric field $\vec{E}$, and the electric polarization $\vec{P}$ due to a point charge q in a linear dielectric medium.
Find the energy density and compare it to the energy density of a point charge in vacuum.

(EM-3):

Find the potential $\vec{A}(\vec{x})$ due to a magnetic dipole at the origin by expanding $1/|\vec{x} - \vec{x}'|$ as a Taylor series in the solution of the Poisson equation $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ for a general localized steady current density $\vec{J}(\vec{x})$, in Coulomb gauge.

(EM-4):

Starting with the differential expression

$$d\vec{B} = \frac{I}{c} \frac{d\vec{l} \times \vec{x}}{|x|^3}$$

for the magnetic induction produced by an increment $I d\vec{l}$ of current, show explicitly that for a closed loop carrying a current I the magnetic induction at an observation point P is

$$\vec{B} = -\frac{I}{c} \vec{\nabla} \Omega$$

where Ω is the solid angle subtended by the loop at the point P.

(EM-5):

A coaxial cable consists of two conducting cylindrical tubes, separated by linear insulating material of magnetic susceptibility χ_m. The current I which is flowing down the tubes in the opposite directions is uniformly distributed over the tubes' surfaces.

Find the magnetic field $\mathbf{B}(\mathbf{r})$, magnetization $\mathbf{M}(\mathbf{r})$, and the bound currents $\mathbf{J}_b(\mathbf{r})$ in the region between the tubes, and $\mathbf{K}_b$ on the tubes' surfaces.

Compute the energy (per unit length) stored in the magnetic field.

Spring 2002 Qualifying Exam

(QM-1):

Consider a potential $V(x) = 0$ for $x > a$, $V(x) = -V_0$ for $0 < x < a$, and $V(x) = \infty$ for $x < 0$.

Show that for $x > a$ the positive energy solutions of the Schroedinger equation have the form $\exp(ikx + 2i\delta) - \exp(-ikx)$.

Calculate the scattering coefficient $|1 - \exp(2i\delta)|^2$ and show that it exhibits maxima (resonances) at certain discrete energies if the potential is suffciently deep and broad.

(QM-2):

A rigid body, with mass M and moment of inertia I (along the z-axis) is constrained to rotate around the z-axis but is free to translate parallel to it.

(1) Write the time-dependent Schroedinger equation for this system,

(2) State the boundary conditions required for the wave functions.

(3) Solve (1) and (2) to explicitly obtain the stationary eigenstates and eigenenergies of the system.

(4) Give examples of degenerate stationary states with definite parity (i.e., also eigenstates of the parity operator $\mathbf{r} \to -\mathbf{r}$), both of even and odd parity.

(QM-3):

Consider an ensemble of composite particles. Each composite particle is made of two particles with angular momentum $J_1 = J_2 = 2$.

(a) If the ensemble is in the pure state $|J = 3, J_z = 2>$ and one measures J_{1z}, what are the possible values and the associated probabilites?

(b) If the ensemble is random and one measures the total angular momentum J, what are the possible values and the associated probabilites in this case?

(QM-4):

Consider a particle in a simple harmonic potential $V(x) = m\omega^2 x^2/2$. At time $t = 0$, the particle is known to be in a linear superposition of the first and second excited states with equal probabilities.

(a) Evaluate the expectation values of x, p, and H at a time t.

(b) Show explicitly that the uncertainty relation does hold in this situation.

(QM-5):

Consider an electron with a magnetic moment $\mathbf{M} = \gamma \mathbf{S}$ in a uniform external magnetic field $\mathbf{B} = B\hat{z}$.

(a) At time $t = 0$ the observer measures S_x and finds the value $+\hbar/2$. Then s/he immediately measures S_y. What is the probability that the value will be $+\hbar/2$?

(b) If a measurement of S_z was done at the time $t = 0$ and the value was found to be $+\hbar/2$, then what are the possible values and the associated probabilities of S_z measured at a later time $t = T$?

(c) A measurement of S_x was done at the time $t = 0$ and the value was found to be $+\hbar/2$, then S_z was measured at the time $t = T$, and, finally, S_z was measured at the time $t = 2T$. What are the possible values and the associated probabilities of the final measurement?

Winter 2002 Qualifying Exam

(EM-1):

Show that it is possible for electromagnetic waves to be propagated in a hollow metal pipe of rectangular cross section with perfectly conducting walls. What are the phase and group velocities? Show that there is a cutoff frequency below which no waves are propagated.

(EM-2):

Inside a superconductor, instead of Ohm's law ($\mathbf{J} = \sigma \mathbf{E}$), we assume London's equations to be valid for the current density $\mathbf{J}$:

$$c\nabla \times (\lambda \mathbf{J}) = -\mathbf{B}, \qquad \frac{\partial}{\partial t}(\lambda \mathbf{J}) = \mathbf{E}$$

(in Gaussian units), and regard λ as a constant. Otherwise, Maxwell's equations (with $\epsilon = 1, \mu = 1$) and the corresponding boundary conditions are unchanged.

Consider an infinite superconducting slab of thickness $2d$ ($-d < z < d$), outside of which there is a given constant magnetic field parallel to the surface:

$$H_x = H_z = 0, \qquad H_y = H_0$$

(same value for $z > d$ and $z < -d$), with $\mathbf{E} = \mathbf{D} = 0$ everywhere. If surface currents and charges are absent, compute $\mathbf{H}$ and $\mathbf{J}$ inside the slab.

(EM-3):

Find a parametric form of the trajectory of a charged particle with a charge q in the perpendicular $\mathbf{E}$ and $\mathbf{H}$ fields of the same magnitude (in V/m and T, respectively).

(EM-4):

A highly relativistic electron e^- (particle 1, rest mass m) collides with a second electron e^- (particle 2, rest mass m) that is at rest in the lab frame prior to the collision. There is just enough energy in the collision (i.e., at threshold) to produce a proton–antiproton pair (each with rest mass M). The reaction is

$$e^- + e^- \rightarrow e^- + e^- + p + \bar{p}.$$

(The two electrons, of course, also emerge from the collision.)
The rest energy of an electron is 0.511 MeV and the rest energy of a proton is 938 MeV. Calculate the total energy in MeV that the relativistic electron must have in order to produce a proton–antiproton pair. Calculate the Lorentz factor of the incident electron.

(EM-5):
In the electric dipole approximation, the angular distribution of radiated power is given by

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}\mathrm{Re}\left[r^2\vec{n}\cdot(\vec{E}\times\vec{B}^*)\right] = \frac{c}{8\pi}k^4|(\vec{n}\times\vec{d})\times\vec{n}|^2,$$

where $\vec{d}$ is the electric dipole moment, which has a harmonic time dependence $[\exp(-i\omega t)]$. Here the wavenumber is $k=\omega/c$ and wave vector is $\vec{k}=k\vec{n}$. Assume that a circularly-polarized plane wave, with complex electric field

$$\vec{E}(t) = E_0\,\vec{e}_+\,e^{-i\omega t},$$

is incident upon an electron at rest. Here, $\vec{e}_+ = (\vec{e}_x + i\vec{e}_y)/\sqrt{2}$. Let the *incident* wave have a wave vector $\vec{k}_{\rm inc}$ that points in the z direction. Use the above radiation formula to calculate the Thomson scattering of this polarized wave from a single electron. Without loss of generality, take the scattered wave vector to lie in the (x, z) plane; i.e.,

$$\vec{k}/k = \vec{n} = \sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_z.$$

Compute the angular distribution of power $\frac{dP}{d\Omega}(\theta)$. Compute the differential cross section $\frac{d\sigma}{d\Omega}(\theta)$ (angular distribution of power divided by incident Poynting flux).

Winter 2002 Qualifying Exam

(QM-1):
Consider an electron in a uniform magnetic field in the positive z-direction. The result of a measurement has shown that the electron spin is along the positive x-direction at $t = 0$. For $t > 0$ compute quantum-mechanically the probability for finding the electron in the state (a) $S_x = 1/2$, (b) $S_x = -1/2$, and (c) $S_z = 1/2$.

(QM-2):
Consider the He atom.
(a) Write the electron Hamiltonian neglecting spin terms.
(b) Assume that the electron-electron interaction can be treated as a perturbation and write explicitly the unperturbed ground state wave function and energy with the electrons treated as distinguishable, one with spin up and the other down.
(c) To first order in perturbation theory, use the results of (a) and (b) to set up an expression for the ground state energy (do not attempt to compute it!).

(QM-3):
Consider two identical particles with mass m and spin $3/2$ in a one-dimensional infinite potential well: $V(x) = 0$ for $|x| < L$ and $V(x) = \infty$ elsewhere.
(a) If there is no interaction between the particles, what is the ground state energy and its degeneracy?
(b) What is the energy of the first excited state and its degeneracy?
(c) Do the ground state and the first excited state have definite parity under parity operation?

(QM-4):
A particle in a one-dimensional simple harmonic oscillator of frequency ω is subjected to a time-dependent, spatially-uniform, force $F(t) = F_0 \cos(\alpha t)$ at $t = 0$. Initially, the particle is in the first excited state. Use first order time-dependent perturbation theory to calculate the probabilities of finding the particle in various eigenstates $|n>$ at time t.

(QM-5):

Consider scattering of a plane wave with momentum $\mathbf{k}$ off a three-dimensional δ-function potential $V(r) = (\gamma\hbar^2/2m)\delta(r - R)$. Use the first order Born approximation to calculate the total scattering cross section as a function of the transferred momentum $\mathbf{q}$.

Winter 2002 Qualifying Exam

(Astro-1):
The equation of transfer involves the determination of the source function,

$$S_\nu = \frac{j_\nu}{\kappa_\nu + \sigma_\nu}. \quad (1)$$

In turn, κ_ν depends on the absorption coefficient per atom, a_ν, and σ_ν is the scattering coefficient.

(a) Write down j_ν. Define your terms.

(b) Assume local thermodynamic equilibrium and detailed balance. Derive the relation between $a_\nu(low \rightarrow up)$ and $\epsilon_\nu(low \rightarrow up)$.

(Astro-2):
The attached figure (Fig.1) depicts the ratio of the specific intensity at various emergent angles, θ, to that at the center of the solar disk. The ratio is obviously a function of wavelength, λ.

(a) What is limb darkening. Describe the cause qualitatively using a sketch.

(b) What physical or atomic/molecular effects cause the dip shortward of 5000 Å and the rise near 1.6 mμ?

(c) Using the gray atmosphere approximation, show how we may use such information to estimate the mass absorption coefficient, κ_λ.

(Astro-3):
Given the simple scaling law between luminosity, $L/L_\odot$, mean molecular weight, μ, the mean mass absorption coefficient, κ_0, and stellar mass, $M/M_\odot$, describe qualitatively the differences in luminosity that arise from the following two situations. In both cases, also describe physically why this happens.

(a) A star with the same chemical composition as the Sun but a mass larger by 50%. What is the ratio of luminosities? Why, physically, is there such a profound difference?

(b) Consider a star with the same mass and heavy element mass fraction as the Sun, but with a helium mass fraction, Y, that is 0.35 compared to the value for the Sun of 0.28. You need not work out the quantitative numbers but show how they would be obtained. Explain physically why the helium mass fraction affects stellar luminosity.

(Astro-4):

Assume a stellar core in a very advanced state of evolution. Assume the conditions are such that the temperature is 5×10^9 K. Pair production is creating enormous numbers of neutrinos.

(a) What is the neutrino energy emission rate in ergs/sec?

(b) What is the thermal energy density?

(c) What is the approximate timescale for this process before it snuffs itself out?

(Astro-5):

Details of big bang nucleosynthesis.

(a) Write out the nuclear reactions involved in the PPI, PPII, and PPIII reaction sequences.

(b) What is the effective energy for the ^{3}He(^{3}He,2p)^{4}He reaction when the temperature T = 10^7 K?

(c) In Big Bang nucleosynthesis, why does the abundance of ^{4}He increase with increasing density?

(d) Why does the abundance of deuterium decline more rapidly with density than does ^{3}He?

Winter 2002 Qualifying Exam

(HEAstro-1):
Assume that a steady wind exists as gas emerges and expands from a confined region. Within the confined source region (e.g., near the surface of a star or from the inner edge of an accretion disk) the gas has negligible initial velocity but is under high pressure p_0 with sound speed a_0. We assume that the gas is a perfect fluid with adiabatic index γ and that the wind flow is adiabatic. (Assume that gravitational forces are negligible.)
Derive the asymptotic wind speed, v_∞.
Does your answer depend upon any assumed geometrical shape of the wind flow? Explain.

(HEAstro-2):
Consider a hypothetical neutron star with a mass of $M = 1.4 M_\odot$. Ignore any residual electrons, protons, and nuclei, and assume that the neutron star is supported primarily by the degeneracy pressure of the neutrons. Assume that equilibrium occurs because the Fermi energy of the neutrons is mildly relativistic.
Give an expression that estimates the radius R of the neutron star under the above simplifying assumptions and compute the numerical value of R.
Note: In CGS units, $1.0 M_\odot = 1.989 \times 10^{33}$ g.

(HEAstro-3):
An astrophysical accretion source is observed to have a continuous spectrum that peaks in the ultraviolet. While it seems likely that several emission mechanisms contribute to the spectrum, a black body component is strongly suggested with a temperature $T_{\rm eff}$ that produces a spectral peak at 20 eV. It has also been ascertained that the absolute bolometric luminosity of the source is $L = 10^{46}$ erg s^{-1}.
Compute the lower bound on the mass of the accreting object.
Compute the effective temperature. Estimate the radius of the thermally emitting region.
The mass and radius can be used to determine the dimensionless strength of the gravitational field. What is this parameter and what is its value here?

Given the strength of the gravitational field, estimate the mass accretion rate $\dot{M}$ (in solar masses per year) that yields L.
Note: In CGS units, $\sigma_T = 0.665 \times 10^{-24}$ cm^2 and $1.0 M_\odot = 1.989 \times 10^{33}$ g.

(HEAstro-4):

Consider a neutron star with a radius of $R = 10.2$ km and a mass of $M = 1.4 M_\odot$. ($M_\odot = 1.989 \times 10^{33}$ g.)
Assume that the neutron star has a constant internal density $\bar{\rho}$. Neglecting general relativistic effects, calculate the functional forms of the pressure profile $P(r)$ and the gravitational potential $\Phi(r)$ inside the star. Determine the numerical value of the pressure P_c at the center of the neutron star.
If hydrostatic equilibrium were disturbed, one possible oscillatory motion would be the f-mode (or breathing mode) in which the neutron star would expand and contract spherically by a slight amount. What characterizes the f-mode is that the oscillation occurs in phase throughout the volume (i.e., there are no nodes within the star such that some regions are contracting while others are expanding). Provide a numerical estimate (and justification) for the period of the f-mode of the neutron star.

(HEAstro-5):
The analysis of the periodic Doppler shifts in the spectrum of HDE 226868 (the companion star of Cygnus X-1) along with the observed orbital period of 5.60 days, leads to a direct observable constraint on the masses of the two stars and on the orbital inclination i. Let M_1 be the mass of HDE 226868 and M_2 be the mass of the compact object. This constraint is the so-called mass function, f, and for Cyg X-1 it has the value $f = 0.25$ M$_\odot$.
Derive the expression for the mass function f.
Optical observations indicate that the supergiant HDE 226868 has an effective temperature of $T = 30,000$ K and a bolometric luminosity of $L = 2.5 \times 10^{38}$ erg s^{-1}, after accounting for distance, extinction and reddening. Spectroscopic analysis and model atmosphere calculations are used to determine that HDE 226868 has a surface gravity of $g_1 = 6.5 \times 10^3$ cm s^{-2}.
Determine the mass M_1 of HDE 226868 **and** determine a bound on the mass M_2 of the compact object.

Spring 2001 Qualifying Exam

(CM-1):
Use Poisson brackets to show that the three components L_i of the angular momentum of a particle are constants of the motion for central forces, i.e. when the Hamiltonian has the form

$$H = \frac{1}{2m}\vec{p}^2 + V(\vec{r}^2).$$

(CM-2):
The Hamiltonian for a particle in a magnetic field $\vec{B}$ is $H = \frac{1}{2m}|\vec{p} - e\vec{A}|^2$ where $\vec{A}$ is the vector potential. For a uniform field in the z-direction this gives (in the Landau gauge)

$$H = \frac{1}{2m}[p_x^2 + (p_y - eBx)^2]$$

a) Write down the Hamilton-Jacoby equation for the Hamilton's characteristic function W;
b) Use the existence of a cyclic variable to write down a separable form for W;
c) Solve for $x(t)$ in term of 3 arbitrary constants
Note: $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$.
d) Find the shape of the orbit.

(CM-3):
Compute the differential cross section $\sigma(\theta)$ for a particle scattering elastically from a hard-sphere target (e.g., a heavy superball) of raduis R.
What is the total scattering cross section? Does the result make intuitive sense?

(CM-4):
Three equal mass points have equilibrium positions at the vertices of an

equilateral triangle. They are connected by equal springs that lie along the arcs of the circle circumscribing the triangle. Mass points and springs are constrained to move only on the circle, so that the potential energy of a spring is determined by the length of the arc.
Determine the eigenfrequencies and eigenvectors (normal modes) of small oscillations of the system. What implications can be drawn by comparing the eigenvalues? Describe the motion of the modes.

(CM-5):
The Hamiltonian for the one-dimensional harmonic oscillator can be written as

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$

a) Determine the value of the constant C such that the following equations define a canonical transformation from the old variables (q, p) to the new variables (Q, P):

$$Q = C(p + im\omega q), \quad P = C(p - im\omega q)$$

b) Obtain the generating function $S(q, P)$ for this transformation;
c) Express the Hamiltonian in terms of the new canonical variables;
d) Write down the Hamiltonian's equations in terms of these new variables and solve the resulting equations. (Note: in quantum theory this transformation defines the creation and annihilation operators).

Spring 2001 Qualifying Exam

(EM-1):
A sphere of raduis R is uniformly charged with the bulk charge density ρ, except for the spherical cavity of a radius r whose center is located at a distance d from the center of the larger sphere ($d + r < R$).
Find the electric field inside the cavity.

(EM-2):
A plane electrode emits electrons with negligible initial velocity in the direction of the opposite electrode. After having been emitted, the electrons accelerate in the electric field created by the two electrodes separated by a distance d. The emission continues until the steady state is achieved where the field of the bulk electric charge compensates the external field at the surface of the emitter: $\frac{\partial\phi}{\partial x}|_{x=0} = 0$.
Find the steady state current $J = v\rho$ (here v is a local charge velocity) as a function of the potential difference V between the electrodes by using the Poisson equation $\Delta\phi = -4\pi\rho$.

(EM-3):
A solid metallic sphere of radius R is placed in a uniform slowly oscillating magnetic field $H(t) = H_0 \cos\omega t$.
Find the power of the dissipated heat if the sphere's conductivity is σ and its magnetic permeability is μ.

(EM-4):
Compute the magnetic scalar potential ϕ_M, where $\vec{H} = -\vec{\nabla}\phi_M$, for a uniformly magnetized sphere with an effective surface magnetic charge density $\sigma_M = \hat{n}\vec{M} = M_0 \cos\theta$, both inside and outside the sphere.

Note:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{t_>^{l+1}} P_l(\cos\gamma)$$

where $cos\dot{\gamma} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$, and $r_<, r_>$ is the set: $|\vec{x}|, |\vec{x}'|$.
Note: $\int_0^\pi d\theta' \sin\theta' \cos\theta' P_l(\cos\gamma) = \delta_{l,1}\frac{2}{3}\cos\theta$.

(EM-5):
Using $\vec{E} = -\vec{\nabla}\phi$, consider a point charge q_1 located at $\vec{x}_1$ relative to the origin. At the origin is a grounded sphere of radius R.
Find the potential $\phi(\vec{x})$ such that $\phi(|\vec{x}| = R) = 0$ on the radius of the sphere, using the method of images.

Spring 2001 Qualifying Exam

(SM-1):

Obtain the pressure of a classical ideal gas as a function of N, V, and T by calculating the partition function.

Obtain the same by calculating the grand partition function.

(SM-2):

In a cavity of blackbody of volume V the density of states of photons over an energy interval $d\epsilon$ is given by $g(\epsilon)d\epsilon = \frac{8\pi V}{c^3 \hbar^3}\epsilon^2 d\epsilon$.

a) Derive the Stefan-Boltzmann law of radiation;

b) Calculate the average number of photons in this cavity at temperature T.

Note: $\int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1} = \pi^4/15$ and $\int_0^\infty \frac{\eta^2 d\eta}{e^\eta - 1} = 2.404$.

(SM-3):

A system of particles is in diffusive contact with a reservoir at temperature T and chemical potential μ. Find the expression of mean-square deviation from the average number of particles $< N >$ in terms of T, μ, and $< N >$.

(SM-4):

A wire of length l and mass per unit length μ is fixed at both ends and tightened to a tension τ.

What is the **rms** fluctuation, in classical statistics, of the midpoint of the wire when it is in equilibrium with a heat bath at temperature T ?

(Note: $\sum_{m=0}^{\infty}(2m+1)^{-2} = \pi^2/8$).

(SM-5):

Consider a sample of an ideal classical gas consising of N identical diatomic molecules of mass M contained in a volume V at temperature T. Each molecule has a permanent electric dipole moment $\vec{p}_0$. A uniform static electric field $\vec{E}$ is applied to the gas.

a) What is the probability that the dipole moment makes an angle θ with

respect to the direction of the applied electric field?
b) What is the electric polarization and the dielectric constant for this gas in the high temperature limit?
c) What is the electric dipole contribution to the heat capacity? Sketch the temperature dependence of this quantity at high T.

Spring 2001 Qualifying Exam

(QM-1):

A particle of mass m is acted upon by the potential $V(z) = mgz$ for $z > 0$ and $V = \infty$ for $z < 0$.

Use the variational wavefunction $\psi(z) = Cz/a$ for $0 < z < a$; $C(2 - z/a)$ for $a < z < 2a$, an 0 elsewhere (after sketching it), to estimate the ground state energy of the particle.

(Note: $< \psi|\psi >= \frac{2}{3}C^2 a$, $\quad < \psi|z|\psi >= \frac{2}{3}C^2 a^2$.

Numerically, for an electron near the Earth's surface $(\hbar^2/m^2 g)^{1/3} = 0.11cm)$.

(QM-2):

Consider two particles with masses m_1 and m_2 moving in one dimension and interacting via

$$V(x_1, x_2) = \frac{k}{2}(|x_1 - x_2| - a)^2$$

where k is a spring constant, and a is their equilibrium separation.

a)Carry out, explicitly, the transformation to the center-of-mass and relative coordinates for the systems' Hamiltonian;

b)Suppose that the lowest energy for the relative motion is much less than $ka^2/2$. In that case, write down the approximate wave function for the system in the relative motion ground state and arbitrary center-of-mass energy.

(QM-3):

A particle of mass m moves in the spherically symmetrical potential $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$.

Find the least value of V_0 such that there is a bound state of zero energy and zero angular momentum.

(QM-4):

The Schroedinger equation for the linear harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n + \frac{1}{2}m\omega^2 x^2 \psi_n = E_n \psi_n$$

with solutions: $\psi_n(x) \propto H_n(x/x_0)\exp(-x^2/2x_0^2)$ where $x_0 = \sqrt{m\omega/\hbar}$, $E_n = (n+1/2)\hbar\omega$.
The first four Hermite polynomials are: $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$.
a) Explain carefully in terms of parity symmetry ($x \to -x$) why the solutions satisfy: $\psi_n(-x) = \pm\psi_n(x)$.
b) What is meant by the statement that a function is an eigenfunction of the parity operator $P : Px = -x$?
c) How does the result in a) correspond to the classical motion of a particle moving in a harmonic potential?

(QM-5):
Rayleigh's expansion for a plane wave along the z-axis in terms of the spherical Bessel functions and Legendre polynomials is

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

Note: $\int_{-1}^{1} P_l(x)P_n(x)dx = 2\delta_{ln}/(2l+1)$.
a)What is the z-component of the angular momentum associated with this wave? Explain.
b)Calculate the probability of finding angular momentum of L units at a given $\vec{r}$, integrated over the polar angles θ and ϕ.
c)This formula was used by Rayleigh in research on sound in the nineteenth century, before quantum mechanics in which it's often used. What's the connection?

Winter 2001 Qualifying Exam

(EM-1):
Calculate the scattering of electromagnetic waves of frequency ω by a small dielectric sphere of radius $\mathbf{R}$ and dielectric constant ϵ. Assume $\mathbf{R} \ll \lambda = c/\omega$ Specifically, find the total cross section up to a numerical factor and a factor inolving ϵ. If you were to guess that factor involing ϵ, what would it be for the case $\epsilon \to 1$?

(EM-2):
The accompanying figure shows a section through the cylindrical plate (radius $\mathbf{b}$) and filament (radius $\mathbf{a}$) of the magnetron. The filament is grounded, the plate is at $\mathbf{V}$ volts positive, and a uniform magnetic field $\mathbf{H}$ is directed along the axis of the cylinder. Electrons leave the filament with zero velocity and travel in curved paths toward the plate. Below what level of V will the current be suppressed by the field $\mathbf{H}$?

(EM-3):
At a given moment of the retarded time t' the velocity of a relativistic particle $\vec{v}$ is parallel to its acceleration $d\vec{v}/dt$. Find the angular dependence of the instant radiation intensity $d\mathbf{I}(t)/d\Omega$, instant total intensity $\mathbf{I}(t)$, and the integral energy loss rate $(-d\mathbf{E}/dt')$.

(EM-4):
A plane polarized electromagnetic wave $\mathbf{E} = \mathbf{E}_i e^{i\mathbf{kr}-i\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega$) having a thickness L. Assuming that in space and in the conducting sheet $\mu = \epsilon = 1$, show

that, except for sheets of very small thickness, the transmission coefficient is approximately equal to

$$T = \frac{32(\mathrm{Re}\beta)^2 e^{-2L/\delta}}{1 - 2e^{-2L/\delta}\cos(2L/\delta) + e^{-4L/\delta}}$$

where $\beta = \sqrt{\frac{\omega}{8\pi\sigma}}(1-i)$ and $\delta = c/\sqrt{2\pi\omega\sigma}$.
(Hint: use the fact that $\beta \ll 1$).

(EM-5):
Show that the force equation $f_\mu = (1/c)F_{\mu\nu}J^\nu$ can be written as

$$f_\mu = \frac{\partial T_\mu^\nu}{\partial x^\nu}$$

where

$$T^{\mu\nu} = \frac{1}{4\pi}[F^{\mu\lambda}F_\lambda^\nu - \frac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}]$$

Winter 2001 Qualifying Exam

(QM-1):

A two-dimensional oscillator has the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(1 + \delta xy)(x^2 + y^2)$$

where $\hbar = 1$ and $\delta \ll 1$. Give the wave functions for the three lowest energy levels for $\delta = 0$; evaluate the first-order perturbation of these levels for $\delta \neq 0$.

(QM-2):

(a) Rotation of an angular momentum eigenvector $|jm, 0>$ through Euler angle β produces new eigenvectors $|jm', \beta>$. These have the property

$$|jm', \beta + 2\pi> = (-1)^{2j}|jm', \beta>$$

What consequence does this property have for half-integer j ($j = 1/2, 3/2, ...$)? How has this been verified experimentally?

(b) Spherical harmonics, $Y_{lm}(\theta, \phi)$, are eigenfunctions in spherical polar coordinates of angular momentum eigenstates with integer $j = l$:

$$Y_{lm}(\theta, \phi) = <\theta, \phi|lm>$$

Explain carefully in terms of operator symmetries why the Y_{lm} must have a definite parity symmetry $Y_{lm}(\pi - \theta, \pi + \phi) = (-1)^l Y_{lm}(\theta, \phi)$ (but you are not to derive this formula).

(c) What is the fundamental reason the angular momentum eigenvalues are quantized, thus $j = 0, 1/2, 1, ...$ only? For example, suppose that quantum mechanics turns out differently in the future, will this property still probably be true? Justify your answers.

(QM-3):

Two identical spin 1/2 particles move in a one-dimensional infinite potential well $V(x) = 0$ if $0 < x < L$ and $V(x) = \infty$ elsewhere.

(a) Write down the ground state wave function and the energy when the two particles are constrained to a triplet spin state.

(b) Do the same for the spin singlet state.

(c) If a very short range repulsive interaction is turned on between particles, discuss qualitatively whether the energy difference between the two states will increase or decrease.

(QM-4):
Consider a particle of mass m moving in a two-dimensional cylindrical potential well $V(r) = 0$, if $r < R$, and $V(r) = \infty$, if $r > R$. Use the trial wave function $\phi(r) = R^2 - r^2$ to estimate the ground state energy.

(QM-5):
Find a total scattering cross section for slow particles of mometum $\mathbf{k}$ in a spherically symmetrical potential $V(r) = -V_0$, if $r < R$, and $V(r) = 0$, if $r > R$.
(Hint: use the Born's approximation and $kR \ll 1$).

University of North Carolina at Chapel Hill

Department of Physics and Astronomy

Part One of Doctoral Written Examination in Physics, 2000

Part B: Statistical Mechanics, Quantum Mechanics I

Friday, May 19, 2000, 1:30 p.m. to 4:30 p.m.

Instructions: Please work in the assigned room, but take a break outside anytime you want to.

Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet, and write <u>only</u> on <u>one side</u> of each sheet.

Identify each sheet by:

Page ____________ of Question ____________ Student # (PID)

SM. Statistical Mechanics

Work 3 out of 5 problems

QM. Quantum Mechanics I

Work 3 out of 5 problems

PARTIAL CREDIT IS GIVEN FOR PARTIAL ANSWERS

My work is completed in full observance of the Honor Code.

______________________________ ______________________________

Signature Print Name

Statistical Mechanics (SM)

(SM-1) The energy levels of a particle of mass M confined in a cubic box of dimension L are described by

$$E_n = \frac{\hbar^2}{2M}\left(\frac{\pi}{L}\right)^2\left(n_x^2 + n_y^2 + n_z^2\right);\ n_x, n_y, n_z = 1, 2, 3, \ldots$$

(A) Derive the partition function Z_N of the ideal gas consisting of N such identical particles.

(B) Show (using Z_N from (A)) that at given temperature T the chemical potential of such ideal gas is given by

$$\mu = k_B T \ln(n/n_Q);\ n = N/L^3 = N/V;\ n_Q = \left(Mk_BT/2\pi\hbar^2\right)^{3/2}$$

Info: $\int_0^\infty exp(-x^2)dx = \frac{\sqrt{\pi}}{2}$; $\ln N! \equiv \frac{1}{2}\ln(2\pi) + \left(N + \frac{1}{2}\right) \ln N - N$

(SM-2) Consider an adsorbent surface having independent adsorption sites. Each of the adsorption sites can be either empty or occupied by one molecule. An adsorbed molecule has energy $-\varepsilon_0$ compared to one in the free state. Show that the coverage ratio θ (number of occupied sites versus the total number of sites), when the adsorbent surface is exposed to an ideal gas at temperature T and pressure P, is given by

$$\theta = \frac{P}{P + P_0(T)} \text{ where } P_0 = \left(\frac{Mk_BT}{2\pi\hbar^2}\right)^{3/2} e^{-\varepsilon_0/k_BT}\ k_BT$$

Info: For ideal gas $\mu = k_BT \ln(n/n_Q)$; $n = N/V$; $n_Q = \left(Mk_BT/2\pi\hbar^2\right)^{3/2}$

(SM-3) If a system is put in an external magnetic field $\vec{h}$, then the Helmholtz free energy is given by $F = U - TS - \vec{m}\cdot\vec{h}$, where $\vec{m}$ is the magnetization, so that $m_z = -\partial F / \partial h_z$ etc. Consider N free and independent spins in an external magnetic field h_z so that the Hamiltonian $H = -h_z\sum_i^N S_{iz}$. Assume $S_{iz} = \pm 1$ only. Show that the magnetization is given by $m = N \tanh\left(\frac{h_z}{k_BT}\right)$.

(SM-4) Derive the density of states $D(\varepsilon)$ as a function of energy ε for a free electron gas in a <u>ONE</u>-dimensional chain of length L. Then calculate the Fermi energy ε_F (at zero temperature) for an N electron system. Use m to denote the mass of an electron. (By the density of states, we mean $N = \int D(\varepsilon)\langle n(\varepsilon)\rangle d\varepsilon$, where $\langle n(\varepsilon)\rangle$ denotes the mean occupation number with energy ε.)

(SM-5) For an ultra-relativistic particle, its energy depends on momentum as $\varepsilon(\vec{p}) = c|\vec{p}|$ where c is speed of light. Consider N such classical non-interacting particles in a box of volume V at temperature T. Put c = 1.

(A) Start with the partition function $Z_N(V,T)$, show that the Helmholtz free energy is given by

$$F = -k_B TN\left[\ln\left(V/\lambda_R^3\right) - \ln N + 1\right],$$

where

$$\lambda_R^{-3} = 8\pi k_B{}^3 T^3/(2\pi\hbar)^3.$$

(B) Determine the equation of state, i.e., how is the pressure related to N, T, and V?

(You may recall the Stirling formula

$$\ln N! \approx N\ln N - N \text{ for large N.}$$

Also you may use

$$\int d^3p\, e^{-\beta p} = 4\pi\int_0^\infty dp\, p^2\, e^{-\beta p} = 4\pi\,\frac{\partial^2}{\partial\beta^2}\int_0^\infty dp\, e^{-\beta p}.)$$

Quantum Mechanics I (QM)

(QM-1) A one-dimensional harmonic oscillator is described by using the operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip_x}{m\omega}\right)$$

(A) Show that the oscillator Hamiltonian is then

$$H = \left(a^+a + 1/2\right)\hbar\,\omega$$

where + denotes Hermitean conjugate.

(B) Suppose we know that

$$a|n> = \sqrt{n}\,|n-1>,\ a^+|n> = \sqrt{n+1}\,|n+1>$$

Derive the following formulas:

$$<n'|x|n> = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

$$<n'|x^2|n> = \frac{\hbar}{2m\omega}\left(\sqrt{n(n-1)}\,\delta_{n',n-2} + (2n+1)\,\delta_{n',n} + \sqrt{(n+1)(n+2)}\,\delta_{n',n+2}\right)$$

Give simple explanations of why $<n|x|n> = 0$ and $<n|x^2|n> = (n+1/2)\,\hbar/(m\omega)$.

(QM-2) (A) Explain why there is a quantum-mechanical uncertainty relation (start with the commutation relation of x and p_x)

$$\Delta x\,\Delta p_x \geq \hbar/2$$

(B) It is often stated in texts on quantum mechanics that the uncertainty relation $\Delta t\,\Delta E \geq \hbar$ follows by analogy with the result in (A). What is wrong with such reasoning?

(QM-3) The Heisenberg equation of motion for operator A and a non-relativistic single-particle Hamiltonian, H, with potential V, is

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A,H]$$

(A) Derive that $\frac{d\langle\vec{p}\rangle}{dt} = -\langle\nabla V\rangle$ and discuss the relation of this Ehrenfest theorem to Newton's equation.

(B) Let $A = \vec{L} = \vec{r} \times \vec{p}$. Show that $\frac{d\langle\vec{L}\rangle}{dt} = \langle -(\vec{r} \times \nabla V)\rangle$ and discuss how this relates to classical mechanics.

(C) The formulas in (A) and (B) contain quantum expectation values, but otherwise correspond to formulas in classical mechanics. Is it necessary to take a classical limit to ensure this correspondence? Explain your answer briefly.

(QM-4) Find the energy eigenvalues and time-independent eigenfunctions for the one-dimensional box potential, where

$$V(x) = \infty \text{ for } x \le 0,\ x \ge L$$
$$V(x) = 0 \text{ for } 0 < x < L.$$

(B) Normalize the eigenfunctions to 1.

(QM-5) If two operators $\hat{A}$ and $\hat{B}$ commute $[\hat{A},\hat{B}] = 0$, then prove that $\hat{A}$ and $\hat{B}$ have a set of nontrivial common eigenfunctions.

University of North Carolina at Chapel Hill

Department of Physics and Astronomy

Part One of Doctoral Written Examination in Physics, 2000

Part A: Classical Mechanics, Electromagnetism I

Friday, May 19, 2000, 9:00 a.m. to 12:00 noon

Instructions: Please work in the assigned room, but take a break outside anytime you want to.

Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet, and write only on one side of each sheet.

Identify each sheet by:

Page ______________ of Question _____________ Student # (PID)

CM. Classical Mechanics

Work 3 out of 5 problems

EM. Electromagnetism I

Work 3 out of 5 problems

PARTIAL CREDIT IS GIVEN FOR PARTIAL ANSWERS

My work is completed in full observance of the Honor Code.

_________________________________ _________________________________

Signature Print Name

Classical Mechanics

(CM-1) Atoms attract each other in a way that can be described by a potential

$$V(r) = -V_0 \frac{r_0^6}{r^6}$$

when r_0 is roughly an atomic radius and V_0 is a constant. Suppose a beam of Hydrogen atoms (mass m) is shot at some Cesium atoms (mass M>>m). A Cesium hydride molecule forms if the atoms ever get closer together than r_0. Find σ_{mol}, the cross section for forming a molecule.

(CM-2) Noether's theorem implies that if the transformations

$$q_i \to q_i + \varepsilon f_i(q)$$

don't change the value of the Lagrangian L, then the quantity

$$X = \sum_i \frac{\partial L}{\partial \dot{q}_i} f_i$$

is conserved. Suppose that a particle of mass m moves in a potential $V(\rho, k\phi + z)$, where ρ, ϕ, z are cylindrical coordinates.

a) Find the transformation that preserves L

b) Find the conserved quantity X

(CM-3) A projectile is shot due east with velocity v_0 and angle α to the horizontal, at latitude ϑ in the northern hemisphere. The Coriolis force has components in the up/down and east/west directions that both affect the range of the projectile (the distance east that it travels before landing). To first order in ω, the rotational velocity of the earth, the effects can be treated separately. To this order find

a) the change ΔR_a in the range due to the vertical component of the Coriolis force

b) the change ΔR_b due to the east/west component

(CM-4) Consider a long thin straight wire of negligible mass suspended in a uniform atmosphere of gas of initial density ρ_0. Suppose a very large electrical current is passed through the wire suddenly, depositing a large amount of energy per unit length σ. The wire explodes and produces, in the central regions, a cylindrical shock wave that expands into the surrounding gas.

Determine answers to the following questions *without* solving the hydrodynamic equations for the detailed solution. Give your reasoning.

a) Determine what powers of time, t^{α}, are exhibited by (*i*) the radius of the shock, (*ii*) the velocity of the shock, and (*iii*) the pressure behind the shock.

b) Building on the results from the first part of the question, give order-of-magnitude correct expressions for the radius of the shock front r, the velocity of the shock front v, and the pressure behind the shock p as functions of time t.

c) Let the gas density be $\rho_0 = 10^{-4}\ g\ cm^{-3}$ and the energy release per unit length be $\sigma = 10^8$ erg cm^{-1}. At a time of one microsecond after the energy release give the approximate radius and velocity of the shock and the pressure behind the shock.

(CM-5) A particle of mass m is constrained to move within the x-y plane. The particle's potential energy, $U(x,y)$, is given by

$$U(x,y) = U_1 \text{ for } x < 0, \text{ and}$$
$$U(x,y) = U_2 \text{ for } x > 0,$$

where U_1 and U_2 are constants. The particle starts in the left half-plane, $x < 0$, with velocity $\vec{v}_1$ and moves into the right half-plane, $x > 0$, where its velocity is $\vec{v}_2$. Find the change in the particle's direction of motion by computing the ratio

$$\frac{\sin\theta_1}{\sin\theta_2},$$

and expressing the result in terms of U_1, U_2, and the initial speed of the particle, $v_1 = |\vec{v}_1|$. Here, θ_1 and θ_2 refer to the angles, on the left and right respectively, between the directions of motion and the x-axis (i.e., the normal to the line marking the discontinuity in the potential energy).

(SM-1):

Obtain the pressure of a classical ideal gas as a function of N, V, and T by calculating the partition function.
Obtain the same by calculating the grand partition function.

(SM-2):

In a cavity of blackbody of volume V the density of states of photons over an energy interval $d\epsilon$ is given by $g(\epsilon)d\epsilon = \frac{8\pi V}{c^3 h^3}\epsilon^2 d\epsilon$.
a) Derive the Stefan-Boltzmann law of radiation;
b) Calculate the average number of photons in this cavity at temperature T.
Note: $\int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1} = \pi^4/15$ and $\int_0^\infty \frac{\eta^2 d\eta}{e^\eta - 1} = 2.404$.

(SM-3):

A system of particles is in diffusive contact with a reservoir at temperature T and chemical potential μ. Find the expression of mean-square deviation from the average number of particles $< N >$ in terms of T, μ, and $< N >$.

(SM-4):

A wire of length l and mass per unit length μ is fixed at both ends and tightened to a tension τ.
What is the **rms** fluctuation, in classical statistics, of the midpoint of the wire when it is in equilibrium with a heat bath at temperature T ?
(Note: $\sum_{m=0}^{\infty}(2m+1)^{-2} = \pi^2/8$).

(SM-5):

Consider a sample of an ideal classical gas consising of N identical diatomic molecules of mass M contained in a volume V at temperature T. Each molecule has a permanent electric dipole moment $\vec{p_0}$. A uniform static electric field $\vec{E}$ is applied to the gas.
a) What is the probability that the dipole moment makes an angle θ with respect to the direction of the applied electric field?
b) What is the electric polarization and the dielectric constant for this gas in the high temperature limit?

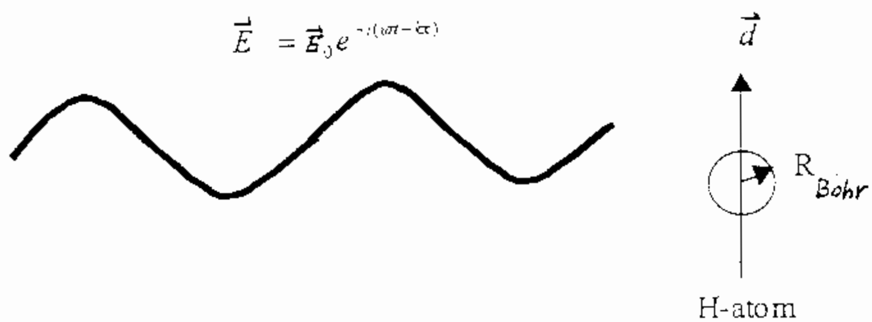
Electromagnetism I

(EM-1) From the microscopic Maxwell's equations in SI units, derive the Biot-Savart expression for the magnetic field $B(\vec{r})$ produced under stationary conditions by a localized, microscopic current density $J(\vec{r})$.

(EM2) Consider a line charge of constant linear charge density λ, along the z-axis between $z = -L$ and $z = L$. Calculate the electrostatic potential $\Phi(\vec{r})$ for all points for which $r > L$, expressed as a series of Legendre polynomial $P_\ell(\cos\theta)$.

(EM-3) A point electric dipole of moment $\vec{p}$ is a distance b from the center of a grounded spherical conductor of radius a, a<b. The dipole points toward the center of the sphere. Calculate the total charge Q on the surface of the sphere, in terms of p, a, and b. Hint: consider the dipole as the limit of two separated point charges.

(EM-4) Rayleigh Scattering of Light by Hydrogen.

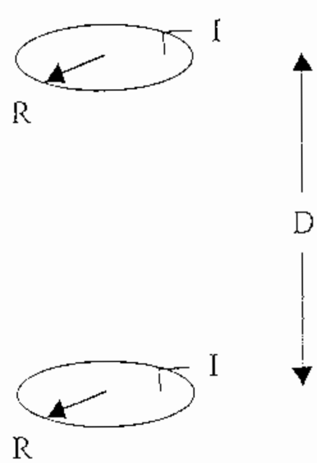


A light wave incident on a hydrogen atom induces an electric dipole momen

$$\vec{d} = \frac{9}{2} R_{BOHR}^3 \vec{E}$$

where $R_{BOHR} = 5.3 \times 10^{-11}$m is the BOHR radius. This dipole moment oscillates with the frequency ω of the wave. At what rate does the atom radiate electromagnetic radiation (energy/time), in terms of E_0, ω, R_{BOHR}, and c?

(EM-5)



Two circular wires of radius R sit in parallel planes, a distance D apart. Each wire carries an electrical current I (of the same sign). What is the force between the wires, in the limit D>>R?

c) What is the electric dipole contribution to the heat capacity? Sketch the temperature dependence of this quantity at high T.

(QM-1):

A particle of mass m is acted upon by the potential $V(z) = mgz$ for $z > 0$ and $V = \infty$ for $z < 0$.
Use the variational wavefunction $\psi(z) = Cz/a$ for $0 < z < a$; $C(2 - z/a)$ for $a < z < 2a$, an 0 elsewhere (after sketching it), to estimate the ground state energy of the particle.

(Note: $< \psi|\psi >= \frac{2}{3}C^2 a, \quad < \psi|z|\psi >= \frac{2}{3}C^2 a^2$.
Numerically, for an electron near the Earth's surface $(\hbar^2/m^2 g)^{1/3} = 0.11cm$).

(QM-2):

Consider two particles with masses m_1 and m_2 moving in one dimension and interacting via

$$V(x_1, x_2) = \frac{k}{2}(|x_1 - x_2| - a)^2$$

where k is a spring constant, and a is their equilibrium separation.
a)Carry out, explicitly, the transformation to the center-of-mass and relative coordinates for the systems' Hamiltonian;
b)Suppose that the lowest energy for the relative motion is much less than $ka^2/2$. In that case, write down the approximate wave function for the system in the relative motion ground state and arbitrary center-of-mass energy.

(QM-3):

A particle of mass m moves in the spherically symmetrical potential $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$.
Find the least value of V_0 such that there is a bound state of zero energy and zero angular momentum.

(QM-4):

The Schroedinger equation for the linear harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n + \frac{1}{2}m\omega^2 x^2\psi_n = E_n\psi_n$$

with solutions: $\psi_n(x) \propto H_n(x/x_0)\exp(-x^2/2x_0^2)$ where $x_0 = \sqrt{m\omega/\hbar}$, $E_n = (n+1/2)\hbar\omega$.

The first four Hermite polynomials are: $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$.

a) Explain carefully in terms of parity symmetry ($x \to -x$) why the solutions satisfy: $\psi_n(-x) = \pm\psi_n(x)$.

b) What is meant by the statement that a function is an eigenfunction of the parity operator P : $Px = -x$?

c) How does the result in a) correspond to the classical motion of a particle moving in a harmonic potential?

(QM-5):

Rayleigh's expansion for a plane wave along the z-axis in terms of the spherical Bessel functions and Legendre polynomials is

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

Note: $\int_{-1}^{1} P_l(x)P_n(x)dx = 2\delta_{ln}/(2l+1)$.

a)What is the z-component of the angular momentum associated with this wave? Explain.

b)Calculate the probability of finding angular momentum of L units at a given $\vec{r}$, integrated over the polar angles θ and ϕ.

c)This formula was used by Rayleigh in research on sound in the nineteenth century, before quantum mechanics in which it's often used. What's the connection?

University of North Carolina at Chapel Hill

Department of Physics and Astronomy

Part One of Doctoral Written Examination in Physics, 1999

Part A: Classical Mechanics, Electromagnetism I

Monday, May 17, 1999, 9:00 a.m. to 12:00 noon

Instructions: Please work in the assigned room, but take a break outside anytime you want to.

Mathematical handbooks and electronic calculators are allowed.

Begin each answer on a new sheet, and write <u>only</u> on <u>one side</u> of each sheet.

Identify each sheet by:

Page ____________, of Question ______ ______ Student # (PID)

CM. Classical Mechanics

Work 3 out of 5 problems

EM. Electromagnetism I

Work 3 out of 5 problems

PARTIAL CREDIT IS GIVEN FOR PARTIAL ANSWERS

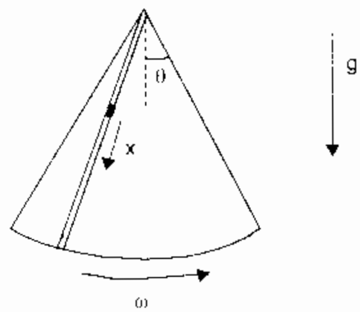
My work is completed in full observance of the Honor Code.

______________________________ ______________________________

Signature Print Name

Classical Mechanics (CM)

(CM-1) A small block is free to slide (under gravity) down a notch along the surface of a cone, which is forced to rotate with angular speed ω. The cone has opening angle θ.



a) Write down the Lagrangian for the block in terms of the generalized coordinate x shown.
b) Determine the conjugate momentum p and write down the Hamiltonian H.
c) Is H the energy? Why or why not?
d) Is energy conserved? Why or why not?

(CM-2) A particle of mass m moves in 2 dimensions under the influence of a potential $V = \frac{1}{2} m \varpi^2 (x^2 + y^2)$, corresponding to an isotropic harmonic oscillator.

a) Find the Hamiltonian of the system.
b) Use the Hamilton-Jacobi equation to solve for x and y as a function of t and 4 arbitrary constants. What is the physical meaning of the generalized momenta you have used?

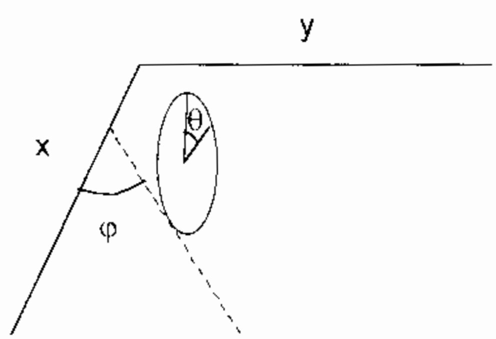
(CM-3) Two pendulums, each of mass *m* and length *l*, are connected at their ends by a massless spring of spring constant *k*. The separation of the pendulums and the equilibrium length of the spring is *d*. For small oscillations about equilibrium:

a) Find the Eigenfrequencies.
b) Find the normal modes (eigenvectors). Describe the motion of the two normal modes.

(CM-4) There is no fuel whose combustion releases energy adequate to raise its own mass out of the Earth's potential well. At one time, this was given as an argument against the possibility of building rockets for space travel. The argument is wrong because it is the rocket, not the fuel, that must be raised off the Earth. The fuel is expelled as a gas to propel the rocket.

a) Write down the equation of motion for a rocket launched vertically in a gravitaitonal field if it is expelling gases with a velocity v'.
b) Integrate this equation to obtain the velocity of the rocket, v, as a function of its total mass m (rocket plus fuel), assuming a constant time rate of mass loss.
c) From energy conservation, derive the formula for escape velocity. Calculate its value for the Earth (ignore air resistance).
d) For a rocket starting from rest, with $v' = 2$ km/s, what is the initial ratio of fuel weight to rocket weight required to reach escape velocity? Assume the rocket expels $1/60^{th}$ of its initial mass during each second of thrust.

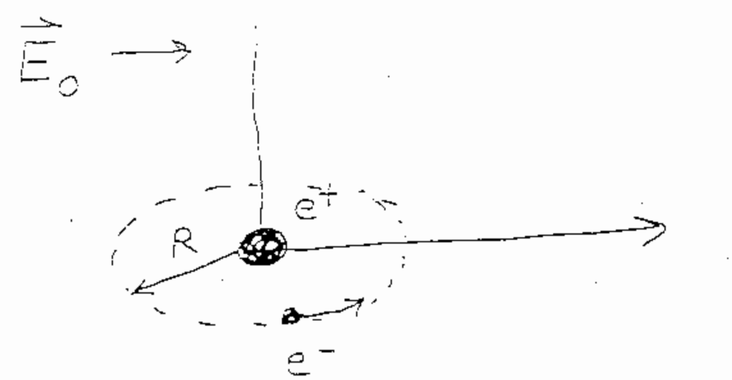
(CM-5) Consider a wheel with mass m, radius R, moment of inertia I_0 around the "axle" axis, and moment of inertia I_1 about an axis through the center of mass in the plane of the wheel. The wheel rolls on a flat surface; it can turn but not tip. Its orientation can be described using 4 coordinates as shown:



a) There are 2 non-holonomic constraints relating $\dot{\theta}, \varphi, \dot{x}$, and $\dot{y}$. What are they?
b) Write the Lagrangian.
c) Write the equations of motion, including the constraints, through 2 Lagrange multipliers.

Electromagnetism I

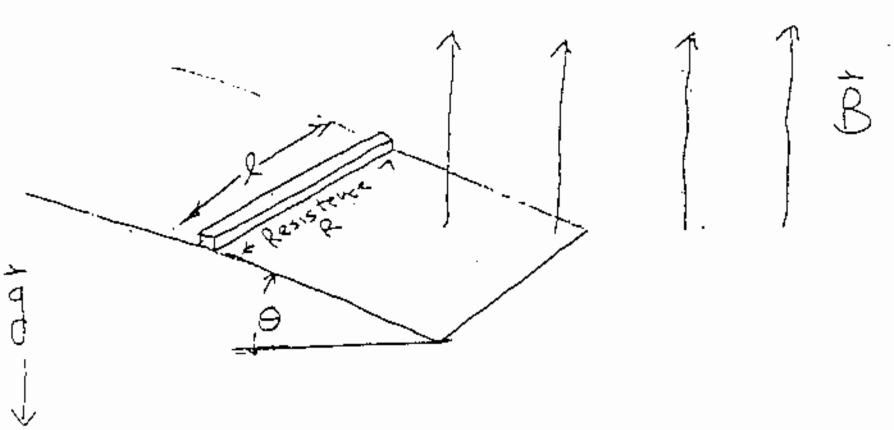
(EM-1) Consider an electron of charge $-e$ moving in a circular orbit of radius, R, around a nuclear charge of $+e$ in an electric field, E_0, directed at right angles to the plane of the orbit. Show that the polarizability, α, of this atom is approximately R^3. (note: consider the positive charge as being infinitely massive so it does not move).



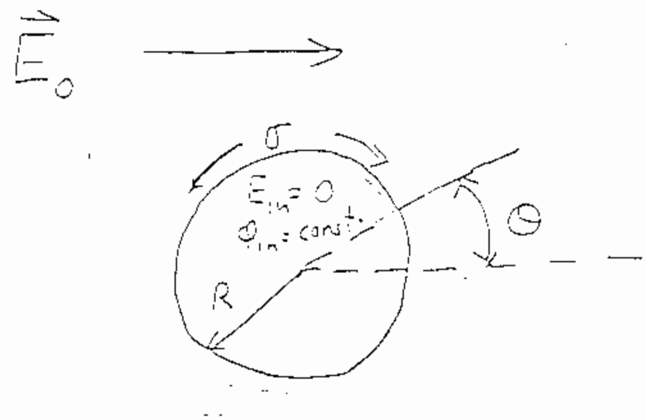
(EM2) A rod of length, l, mass, m, and electrical resistance, R, is free to slide down frictionless, parallel conducting rails. These rails are also superconductors so their electrical resistance is 0. The sliding rod therefore closes a conducting loop as shown below. The plane of the rails makes an angle, θ, with the horizontal and a uniform vertical magnetic field, B, throughout the region.

a) Find the terminal velocity of the sliding rod in terms of l, m, θ, B and the local acceleration due to gravity, g.

b) Show that the rate of resistive heating in the rod is equal to the rate at which the rod is losing gravitational potential energy.



(EM-3) A uniform electric field, E_0 is to be completely shielded from the interior of a sphere of radius, R, by placing a charge layer, σ, on its surface. Find the distribution of the charge (the geometry is shown below).

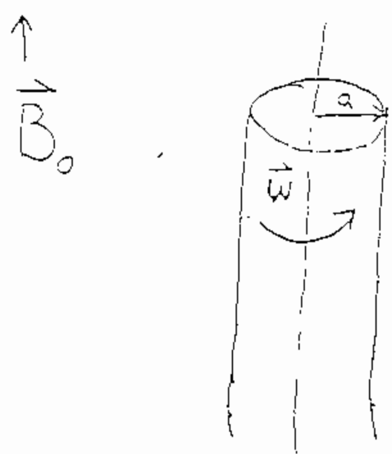


(EM-4) An infinitely long solid metal cylinder rotates about its axis with constant angular velocity, w. A uniform magnetic field, B_0, exists parallel to the cylinder axis. Obtain the following quantities

a) Charge density inside the cylinder

b) Surface charge density

c) The electric field and the scalar potential



(EM-5) A point charge, q, is a distance, d, away from an infinite plane conductor held at zero potential. Use the method of images to find:

a) the surface charge density on the plane

b) the force between the plane and the charge

c) the work necessary to remove the charge from its position to infinity

d) the potential between the charge q and its image (compare with answer to (c))