The nucleus as a condensate of collective quark triplets

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Starting from an effective hamiltonian for multi-quark systems, we propose a mapping of the quark degrees of freedom onto collective colorless triplets of constituent quarks. We end up with an effective hamiltonian for nucleons, which can be treated by traditional nuclear many-body approximations. In this way, we can describe subnucleonic degrees of freedom in the quantum liquid of nucleons.

Traditionally, the nucleus is treated as a quantum liquid of pointlike structureless nucleons. More recently, constituent models of the nucleon have called this simple picture into question. The relatively large size of the nucleons in such models and a number of recent experiments (by, e.g., the EMC) indicate that the nucleon and nuclear scales do not completely decouple.

As a consequence, there is currently great interest in trying to describe multinucleon systems directly in terms of their QCD constituents. Ideally, the treatment should be fully relativistic and should include quark, gluon and antiquark degrees of freedom, but to date there has been little progress in this direction. In contrast, significant progress has been made in the use of nonrelativistic constituent quark models, which in principle can be obtained from QCD by eliminating the gluon and antiquark degrees of freedom. For example, the quark cluster model [1,2] has been used with great success to derive properties of two-baryon systems.

Efforts to apply the constituent quark model to many-nucleon systems have not been as successful. The major obstacle is colorless three-quark clustering, which cannot be implemented with traditional many-body methods. So far only two-body clustering can be treated in a natural way, through the use of the BCS approximation.

In this letter, we propose a method for incorporating colorless three-quark clustering in a microscopic treatment of many-nucleon systems. Our method is based on the use of iterative mapping techniques [3]. We will describe its application to several relatively simple model hamiltonians for interacting quarks, introduced by Petry and collaborators [4]. However, our method is by no means restricted to such simple model hamiltonians, but can be applied in principle to any effective quark interaction, possibly including the three-quark terms that are expected to arise from the three-gluon exchange diagram of QCD.

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As is well known, three-fermion clusters are not exact fermions. Our method maps colorless three-quark clusters onto triplet fermions that *do* satisfy exact anticommutation rules. The mapping leads from the underlying multi-quark interaction to an effective hamiltonian for these triplet fermions, which can be treated by the familiar many-body techniques of conventional nuclear structure physics. This effective triplet hamiltonian incorporates the physics of the Pauli principle at the quark level. The nuclear ground state in such an approach represents a quantum liquid of composite particles, the nucleons. Higher-lying states involve both excitations within the space of the nucleons and excitations of the nucleons themselves.

As in traditional boson mapping procedures [5], the mappings are implemented by requiring that the commutation algebra of bilinear operators in the mapped (triplet-fermion) space be the same as in the original quark space. The effects of the Pauli principle are then transmitted to the mapped operators, which as a consequence either involve infinite series expansions or are non-hermitean. In the Dyson approach, which we use throughout, the resulting triplet-fermion hamiltonians are non-hermitean but of finite order.

The details of the mapping depend on the underlying model. As noted earlier, we will describe its application to two models due to Petry and coworkers. In these models, quarks interact via a simple phenomenological separable interaction, which acts only between pairs of quarks coupled to spin and isospin zero. In the first and simplest model (which we refer to as the single-orbit model) the quarks are restricted to a single-j orbit; we also discuss a more complicated version (the multi-orbit model) in which they can occupy several j orbits. The single-orbit model can be solved exactly by group theoretical methods, as can the multi-orbit model when the single-quark energies are degenerate. When they are not degenerate, however, the model cannot be solved exactly. The non-degenerate model has some attractive features and has been put forth as a semi-realistic quark model of the nucleus. By introducing single-quark energies appropriate to a confining bag whose radius is that of the whole nucleus, one can derive from it a nuclear shell model. However, as has been shown by Suzuki and Hecht [6], this model is unable to produce spatially localized nucleons, a necessary criterion for a truly realistic quark model. Other arguments against the model were given in ref. [7]. For our purposes, the validity of the Petry model is not an issue; we simply use it to test our proposed method.

We begin by applying our method to the single-orbit model, since it is the simplest and most clearly illustrates the basic ideas. The fundamental operators in the model are the unpaired quarks $q_l^\dagger = q^\dagger_{mt}$, with quantum numbers 1, m and t labeling color, j_z and isospin, respectively, and the bilinear quark operators

$$A_1^\dagger = \sum_{23mt} \epsilon_{123} q^\dagger_{2mt} q^\dagger_{3\tilde{m}\tilde{t}}, \quad A_1 = (A_1^\dagger)^\dagger ,$$

$$J_{11'} = \sum_{mt} q^\dagger_{mt} q_{1'mt}, \quad \hat{N} = \sum_1 J_{11} , \tag{1}$$

with $q^\dagger_{\tilde{m}\tilde{t}} = (-)^{j-m+1/2-t} q^\dagger_{-m-t}$. The bilinear quark operators form a closed algebra:

$$[A_1, A_2^\dagger] = 4[\delta_{12}(2\Omega - \hat{N}) + J_{12}] , \quad [J_{11}, A_1^\dagger] = 0 ,$$

$$[J_{21}, A_1^\dagger] = -A_1^\dagger , \quad [J_{11}, A_2^\dagger] = A_2^\dagger , \quad [J_{12}, A_2^\dagger] = 0 \quad (1 \neq 2) \tag{2}$$

with $\Omega = 2j+1$. The hamiltonian of the model has the form

$$H = -\tfrac{1}{4} G \sum_1 A_1^\dagger A_1 . \tag{3}$$

We carry out the mapping in two steps. In the first, we map the original quark space using the Dyson method onto a space consisting of bosons $B_{lk}^\dagger$ and ideal fermions $a_l^\dagger$ which by definition commute with the bosons:

$$q_l^\dagger q_k^\dagger \Rightarrow B_{lk}^\dagger - \sum_{ll'} B_{ll}^\dagger B_{kl'} B_{ll'} - \sum_l (B_{ll}^\dagger a_k^\dagger a_l - B_{kl}^\dagger a_l^\dagger a_l)$$

$$q_k q_i \Rightarrow B_{ik} ,$$

$$q_l^\dagger q_k \Rightarrow \sum_l B_{il}^\dagger B_{kl} + a_l^\dagger a_k . \tag{4}$$

This mapping preserves the commutation relations (2). Introducing collective bosons

$$B_1^\dagger = \frac{1}{\sqrt{2\Omega}} \sum_{123mt} \epsilon_{123} B^\dagger_{2mt,3\tilde{m}\tilde{t}} , \tag{5}$$

and truncating for the moment to the associated collective subspace (the hamiltonian (3) separates it exactly from the rest of the space), we find for the Dyson image of the hamiltonian

$$H_c = -2\Omega G\hat{N}_B + G\hat{N}_B(\hat{N}_B - 1) - H_{int} , \quad (6)$$

where $\hat{N}_B$ is the number operator for bosons and H_{int} involves an interaction between fermions and bosons:

$$H_{int} = G \sum_{1 \neq 2, mt} (B_1^\dagger a_{1mt}^\dagger B_2 a_{2mt} - B_1^\dagger B_1 a_{2mt}^\dagger a_{2mt}) . \quad (7)$$

In the second step[#1], we map boson–fermion pairs $B_{ik}^\dagger a_l^\dagger$ onto triplet fermions $c_{ik,l}^\dagger$, which are defined to obey the anti-commutation relations

$$\{c_{ik,l}, c_{i'k',l'}^\dagger\} = \delta_{ll'}(\delta_{ii'}\delta_{kk'} - \delta_{ik'}\delta_{ki'}) . \quad (8)$$

Using the method of Dyson again, we obtain for the general mapping of boson–fermion pairs to triplet fermions

$$B_{ik}^\dagger a_l^\dagger \Rightarrow c_{ik,l}^\dagger - \sum_{p<q,r} c_{ik,r}^\dagger c_{pq,l}^\dagger c_{pq,r} ,$$

$$B_{ik} a_l \Rightarrow c_{ik,l} ,$$

$$a_l^\dagger a_k \Rightarrow \sum_{p<q} c_{pq,i}^\dagger c_{pq,k} ,$$

$$B_{ii'}^\dagger B_{kk'} \Rightarrow \sum_l c_{ii',l}^\dagger c_{kk',l} . \quad (9)$$

Truncating to fully antisymmetric, collective (colorless) triplet fermions (again separated by the hamiltonian)

$$c_{mt}^\dagger = \frac{1}{\sqrt{3}} \sum_{123m't'} \epsilon_{123} c_{1m't'}^\dagger c_{2\bar{m}'\bar{t}',3mt} , \quad (10)$$

we arrive at the hamiltonian

$$H_c = -G(2\Omega+3) \sum_{mt} c_{mt}^\dagger c_{mt} + G \sum_{mtm't'} c_{mt}^\dagger c_{mt} c_{m't'}^\dagger c_{m't'} , \quad (11)$$

which has eigenvalues

$$E(N_c) = -2GN_c(\Omega+1) + GN_c(N_c-1) , \quad (12)$$

where N_c is the number of collective triplets. This corresponds precisely to the eigenvalues that were found by Petry for the $(l=0)$-sector, where the quantity l corresponds to the number of triplets in which two of the particles are not paired. The $(l=1)$-sector, for instance, corresponds to (N_c-1) nucleons and one Δ-particle. In deriving eq. (11), we dropped these configurations by neglecting the non-collective bosons in eq. (6).

One can easily include these non-collective bosons in our treatment by supplementing the collective pair algebra of eq. (5) with non-collective pairs. The result is to add to the hamiltonian (11) an additional term, which couples nucleons to Δ-particles:

$$H_{cd} = 2G\hat{N}_c\hat{N}_d , \quad (13)$$

where $\hat{N}_d$ is the number operator of non-collective triplets (i.e. those that involve non-collective pairs). The eigenvalues of the full hamiltonian can then be written as

$$E(N_c, N_d) = -G[(N_c+N_d)(2\Omega+3-N_c-N_d) - N_d(2\Omega+3-N_d)] , \quad (14)$$

which exactly reproduces the spectrum of Petry, when there are non-collective pairs. Thus, our iterative mapping method is exact for the single-orbit model.

Next we consider the multi-orbit model, for which the hamiltonian is

$$H = \sum_{1jmt} \varepsilon_j q_{1jmt}^\dagger q_{1jmt} - \tfrac{1}{4}G \sum_{1jj'} A_{1j}^\dagger A_{1j'} , \quad (15)$$

where

$$A_{1j}^\dagger = \sum_{23mt} \epsilon_{123} q_{2jmt}^\dagger q_{3j\bar{m}\bar{t}}^\dagger \quad (16)$$

is the collective pair for orbit j. Using the same two-step iterative mapping method as above, and restricting to the single collective $J=0$, $T=0$ pair for each orbit, we end up with colorless triplet–fermion operators $c_{j',jmt}^\dagger$, in which two particles in orbit j' are fully paired and the third in orbit j is unpaired. The resulting hamiltonian expressed in terms of these partially-correlated triplet–fermion operators has the form

[#1] A similar method has been discussed by Zhu Yao-yin et al. [8], in the context of the composite particle representation method and applied to the one-baryon system.

$$H=\sum_{j_1 j_2 jmt}\Bigg((\varepsilon_{j_1}+\varepsilon_{j_2}+\varepsilon_j)\delta_{j_1 j_2}-2G\sqrt{\Omega_{j_1}\Omega_{j_2}}$$

$$-2G\sqrt{\frac{\Omega_{j_2}}{\Omega_{j_1}}}\delta_{jj_1}\Bigg)c^{\dagger}_{j_1,jmt}c_{j_2,jmt}$$

$$-G\sum_{jj_1}\sqrt{\frac{\Omega_{j_1}}{\Omega_j}}\sum_{j_2 j_3}\sum_{mm'tt'}(c^{\dagger}_{j,j_2mt}c^{\dagger}_{j,j_3m't'}c_{j_1,j_2mt}c_{j,j_3m't'}$$

$$+\tfrac{2}{3}c^{\dagger}_{j_2,jmt}c^{\dagger}_{j,j_3m't'}c_{j_2,j_3m't'}c_{j_1,jmt}$$

$$-\tfrac{2}{3}c^{\dagger}_{j_2,jmt}c^{\dagger}_{j,j_3m't'}c_{j_1,j_3m't'}c_{j_2,jmt})\,. \qquad (17)$$

This hamiltonian clearly exhibits the non-hermitean nature of the Dyson method. In addition it is no longer analytically solvable except in the limit where all single-particle energies ε_j are degenerate. However, it can be treated by conventional many-body approximation techniques.

In the specific test applications to follow, we will focus on the nucleus ^{16}O. In a doubly-magic system it is natural to use the (non-unitary) spherical Hartree–Fock (HF) method, wherein the ground state is described by Slater determinants built up from single-particle states with definite angular momenta. For a non-hermitean hamiltonian, the bra and ket determinantal state vectors need not be adjoints of one another. Thus, we introduce separate bra and ket states:

$$|\Psi\rangle=\prod_{i=1,\ldots,N}\Gamma^{\dagger}_{j_i m_i t_i}|0\rangle\,,$$

$$\langle\Phi|=\langle 0|\prod_{i=1,\ldots,N}\gamma_{j_i m_i t_i}\,. \qquad (18)$$

The correlated creation and annihilation operators, $\Gamma^{\dagger}_{jmt}$ and γ_{jmt}, are linear combinations of the fundamental single-particle creation and annihilation operators:

$$\Gamma^{\dagger}_{jmt}=\sum_{j'}X_{j'j}c^{\dagger}_{j',jmt}\,,$$

$$\gamma_{jmt}=\sum_{j'}Y_{j'j}c_{j',jmt}\,. \qquad (19)$$

Their structure coefficients are determined by the variational condition

$$\delta\frac{\langle\Phi|H|\Psi\rangle}{\langle\Phi|\Psi\rangle}=0\,, \qquad (20)$$

which can be solved with standard iterative techniques. Interestingly, the structure coefficients of the various correlated triplet fermions (i.e. nucleons) need not be the same.

We have carried out several calculations for the nucleus ^{16}O with different parameter choices in the multi-orbit model. In each case, we assumed that the quarks were confined by a harmonic oscillator potential (with oscillator energy $\hbar\omega$) and restricted to the orbits of the 1s, 1p and 2s–1d oscillator shells. The unpaired quarks were assumed to fill the 1s and 1p shells only.

The first calculation assumed a pairing strength of $G=7$ MeV and degenerate single-particle orbits. With the above choice of G, the lowest non-nucleonic excitations occur at a reasonable energy of roughly 280 MeV. The results of this calculation are given in table 1. We show both the calculated ground state energy and the occupation numbers for the various quark orbits. With degenerate orbits, the multi-orbit model has the same group structure as the single-orbit model and thus, as noted earlier, can be solved exactly. Indeed, the eigenvalue expression is identical to that of the single-orbit model (12) except that one must identify $\Omega=\sum_j\Omega_j$. For $N=16$ and $\Omega=40$, the analytic result is $E_{GS}=-3024$ MeV, which is reproduced exactly by the spherical Hartree–Fock calculation. The occupation numbers reflect the assumption that the

Table 1
Spherical HF results for ^{16}O using the multi-orbit Petry model. Calculations are carried out for degenerate single-quark energies and for a pairing strength of $G=7$ MeV. Shown are the number of quarks in the active quark orbits and the ground state energy.

G (MeV)	Quark occupation numbers						E_{GS} (MeV)
	$1s_{1/2}$	$1p_{3/2}$	$1p_{1/2}$	$1d_{5/2}$	$2s_{1/2}$	$1d_{3/2}$	
7	7.429	14.857	7.429	9.143	3.048	6.095	−3024.00

Table 2
Spherical HF results for ^{16}O using the multi-orbit Petry model. Calculations are carried out for an oscillator energy $\hbar\omega = 16$ MeV and for three values of the pairing strength G. Shown are the number of quarks in the active quark orbits and the ground state energy.

G (MeV)	Quark occupation numbers						E_{GS} (MeV)
	$1s_{1/2}$	$1p_{3/2}$	$1p_{1/2}$	$1d_{5/2}$	$2s_{1/2}$	$1d_{3/2}$	
0	36	8	4	0	0	0	192.00
1.4	10.69	17.25	8.63	5.72	1.91	3.81	254.02
7	8.13	15.31	7.66	8.45	2.82	5.63	−2098.90

unpaired quarks fill the 1s and 1p shells. Because of the degeneracy in single-quark energies, the unpaired quarks could have been distributed in any other fashion with no change in energy.

We next carried out a series of calculations for nondegenerate single-quark energies appropriate to $\hbar\omega = 16$ MeV, which is reasonable for the nucleus ^{16}O (see table 2). These calculations had two purposes. The first was to illustrate that approximate solutions could indeed be obtained even when the orbits are nondegenerate. The second was to see whether our HF solutions violate the Pauli principle at the underlying quark level. Pauli violations have historically been a major problem in mapping techniques. If the mapped hamiltonian could be diagonalized exactly, there would be a clean separation between physical states that properly reflect the underlying Pauli principle and unphysical states that do not. However, once approximations are made (as invariably they must be) physical and unphysical states can mix and as a consequence lead to Pauli violations. We see from the table that for $G=0$, the (trivial) HF solution indeed produces Pauli violations. More specifically, there are more quarks in the lowest $1s_{1/2}$ quark orbit than the 12 permitted by the Pauli principle. In the most realistic scenario ($G=7$ MeV), however, this is not the case. Furthermore, even when we reduce the pairing strength from this value by a factor of five, we still find no Pauli violations at the quark level. The reason for this is the following: There are two fundamental scales in our problem, the nuclear scale (which is governed by the single-particle energies and is typically of the order of MeV), and the nucleon scale (which is governed by $G\Omega$ and is of the order of hundreds of MeV). The two-quark interaction, which produces di-quark clustering, works at the nucleon scale. It thus scatters quarks strongly over a large number of single-quark orbits (which were produced at the nuclear scale) and this strong scattering precludes Pauli violations. Any reasonable quark interaction will have to produce such strong clustering at the nucleon scale. Thus, we expect that unphysical states and Pauli violations will not prove a problem in subsequent realistic applications of our method.

For the future, there are several formal issues still to be investigated. We need to consider the application of our method to more general quark hamiltonians, which include not only more realistic two-quark interactions but also three-quark forces. As noted by Suzuki and Hecht [6], three-quark interactions (suggested by QCD) may be necessary to spatially localize colorless quark triplets. In the case of more general hamiltonian, the commutation algebra is more difficult to preserve. In general, we must not only preserve the algebra of bilinear operators, as we have done in the present work, but we must also include the single-quark creation and annihilation operators in order to guarantee a physical subspace. Indeed, already in our nondegenerate Petry model calculations, there exist states in the original quark space that do not seem to appear properly in the mapped space. However, such states most likely appear very high in energy and should not influence the ground state significantly. Improper treatment of such states could, however, be of greater significance when more general interactions are considered. One possible way to properly include the single-quark operators in the algebra is through the use of the quantized Bogolyubov transformation [9]. In addition, we will most likely have to consider the generalization of other traditional many-body techniques [10] (e.g. TDA, RPA and Brueckner theory) to multi-index fermions interacting via non-hermitean hamiltonians, which invariably arise in Dyson mappings.

One these formal questions have been adequately resolved, we hope to begin addressing a variety of interesting issues. At the level of traditional nuclear phenomena, we should be able to calculate such quantities as the magnetic moments of single-particle or single-hole nuclei (the so-called Schmidt moments) directly from quarks. Indeed, Arima et al. [11] have shown that the Petry pairing hamiltonian cannot reproduce the Schmidt moments. They suggest, however, that including other QCD-motivated terms in the quark hamiltonian might help. We also hope to address several issues of more contemporary interest, such as the structure of nucleons in the nuclear medium (as raised, for instance, by the EMC-effect) and the role of nucleonic excited states in nuclear structure.

In summary, we have described a new approach to deriving the properties of nuclei from the interactions between its constituent quarks. Through the use of iterative mapping techniques, we are able to go from quarks to colorless triplet fermions. These are true fermions, satisfying exact anticommutation rules, and thus they can be treated with familiar fermion many-body approximation techniques (albeit slightly generalized). They interact via an effective hamiltonian that fully incorporates the underlying quark Pauli effects. Variational treatments of many-nucleon systems will pick out special collective triplet fermions which can be identified as nucleons. Effects associated with excited states of the nucleon can be readily treated in this fully consistent framework.

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