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# INTERACTING BOSON MODEL OF COLLECTIVE OCTUPOLE STATES (I). The rotational limit

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**Abstract:** We discuss the problem of describing low-lying collective negative-parity states within the framework of the interacting boson model. We suggest that a simultaneous description of quadrupole and octupole states in nuclei be done in terms of the group U(16), which includes f- and p-bosons in addition to the usual d- and s-bosons. We analyze the dynamical symmetries associated with the rotational limit of this model and discuss their classical (large-$N$) analogs. We conclude with a preliminary application of the model to the radium isotopes.

## 1. Introduction

Low-lying collective nuclear states are dominated by the occurrence of quadrupole vibrations and deformations. Their properties can be described in terms of shape variables, $\alpha_{2\mu}$ ($\mu = 0, \pm 1, \pm 2$), ref [1]) or alternatively in terms of interacting s- and d-bosons with $J^P = 0^+$ and $J^P = 2^+$ respectively [2]). The role played by d-bosons is easily understood since they can be thought of as a quantization of the variables $\alpha_{2\mu}$. The introduction of s-bosons is less obviously necessary and arose from a study of the underlying microscopic structure which led to an interpretation of bosons in terms of nucleon pairs [3]). It reflects the existence in nuclei of a pairing interaction in addition to the quadrupole interaction. An important consequence of the introduction of s-bosons is that it facilitates phenomenological descriptions of spectra. For example, with s- and d-bosons it is straightforward to obtain an SU(3) symmetry that describes accurately quadrupole deformations.

Although quadrupole degrees of freedom dominate the low-lying features of nuclei, other degrees of freedom may play some role. The next multipolarity is the octupole, introduced into nuclear physics early on by Bohr and Mottelson [4]) through the shape variables $\alpha_{3\mu}$ ($\mu = 0, \pm 1, \pm 2, \pm 3$). In this article, we discuss how to describe octupole degrees of freedom in terms of interacting bosons. It is obvious that one needs here $J^P = 3^-$ (octupole or f) bosons. As long as only octupole vibrations are considered the introduction of only f-bosons appears to be sufficient to describe data; recently Barfield [5]) analyzed spectra of several nuclei in the rare-earth region

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using a model consisting of s-d bosons plus one f-boson. However, there is growing evidence that octupole deformation may also occur in nuclei [6]). One important question that has arisen recently is how best to describe these deformations. This problem has been addressed in the framework of Bohr-type hamiltonians by Rohozinski [7]). We discuss here how to address the problem in terms of interacting bosons. We suggest that in general octupole deformed nuclei be described by a system consisting of $J^P = 1^-$ and $J^P = 3^-$ (p and f) bosons in addition to the usual s- and d-bosons. The introduction of p-bosons here is similar to that of s-bosons in the description of quadrupole deformation in that it both facilitates the phenomenological treatment, and appears to be dictated by the underlying microscopic structure [8,9]).

When p- and f-bosons are introduced alongside s and d, the space spanned by single boson states becomes $1+3+5+7=16$ dimensional. The corresponding algebraic structure is U(16). This structure is rather large and when treated as a phenomenological model it contains many adjustable parameters. We find it convenient then to study the dynamic symmetries of this structure. As in the corresponding case of quadrupole vibrations and deformations, dynamic symmetries provide appropriate benchmarks for phenomenological descriptions. In this article we describe the dynamic symmetries of the U(16) model associated with rotational spectra. Using these symmetries as starting points, we then analyze numerically spectra and transitions in light actinide nuclei. A similar, but simpler, numerical analysis of the role of p-bosons in octupole vibrational states has been presented recently by Han *et al.* [10]).

## 2. The model

The single boson states contained in U(16) are depicted schematically in fig. 1. Since, in a microscopic interpretation, bosons are related to fermion pairs in shell-model states [3]), the appropriate counting of the boson number $N$ is here more complicated than in the case of U(6). While s- and d-bosons have their origin largely in valence shell ($0\hbar\omega$) pairing, f and especially p bosons must include excitations across the major shells ($1\hbar\omega$). We will consider here, for simplicity, the case in which both protons and neutrons are particle-like. The number $N$ will then be taken as the total number of proton, $N_\pi$, and neutron, $N_\nu$, pairs outside the doubly closed shells, $N = N_\pi + N_\nu$. For example, in $^{200}_{88}Ra_{132}$, $N_\pi = 3$, $N_\nu = 3$, $N = 6$. This counting implies that both sd and pf bosons are considered to be nucleon pairs. The situation is shown schematically in fig. 2 where the pair states(bosons) corresponding to 6 particles (either protons or neutrons) are shown.

A question that immediately arises is to what extent p bosons represent "spurious" center-of-mass excitations. It appears [8]) that even when the spurious excitations are removed, a moderately collective pair state remains. Without the removal of the center-of-mass motion, the p-state would be very collective and in fact appear at

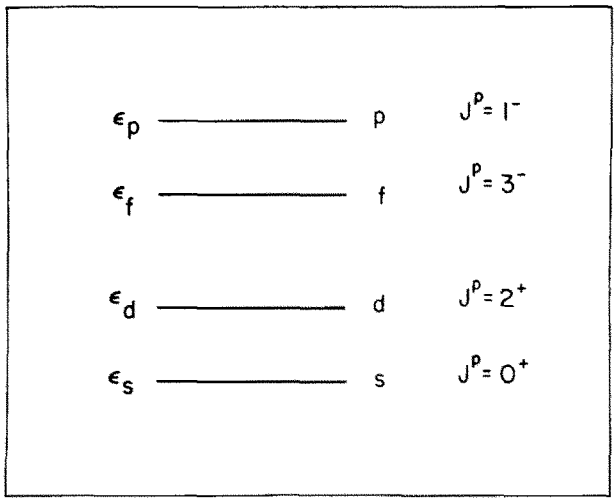


Fig. 1. Schematic representation of the boson states included in the U(16) model.

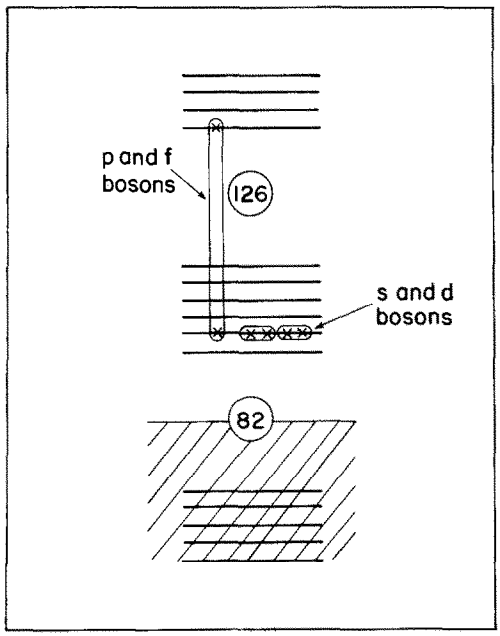


Fig. 2. Schematic representation of the pair states that provide a microscopic interpretation of the U(16) model. The example shown in the figure corresponds to 6 particles described as $N = 3$ bosons, 2 sd bosons and 1 pf boson.

zero energy in an exact calculation. Removing the center-of-mass motion results in a higher energy for the p-boson, as depicted schematically in fig. 3. This may reflect the fact that the p-state is collective not with respect to the single particle operator $rY_{1\mu}$, but rather to the operator $r^3 Y_{1\mu}$. The $p$-collectivity is then similar to that of the octupole operator $r^3 Y_{3\mu}$.

From the phenomenological point of view the microscopic origin of the bosons (although an interesting and important problem) is not relevant as long as one specifies the total boson number $N$. We henceforth consider the phenomenology of a system of $N$ s, p, d, f bosons. In general, the hamiltonian for U(16) will contain a part describing s- and d-bosons, a part describing p- and f-bosons and a part describing their interaction

$$H = H_{\mathrm{sd}} + H_{\mathrm{pf}} + V_{\mathrm{sd,pf}} . \tag{2.1}$$

This hamiltonian, as well as the transition operators, can be written in terms of boson creation and annihilation operators,

$$\begin{array}{lll} s^\dagger & s & \\ d^\dagger_\mu & d_\mu & \mu = 0, \pm 1, \pm 2 \\ p^\dagger_\mu & p_\mu & \mu = 0, \pm 1 \\ f^\dagger_\mu & f_\mu & \mu = 0, \pm 1, \pm 2, \pm 3 . \end{array} \tag{2.2}$$

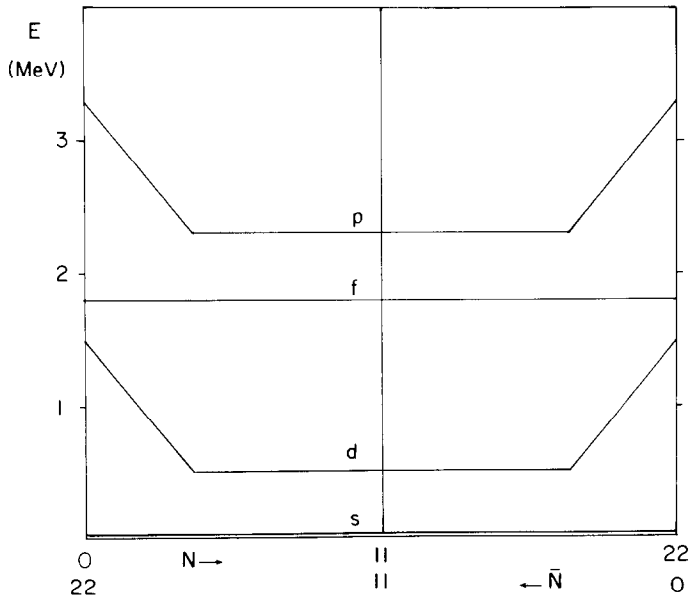


Fig. 3. Schematic representation of the expected single boson energies in a major shell. $N$ is the number of boson particles and $\bar{N}$ the number of boson holes.

The hamiltonian will contain all bilinear and quadrilinear terms that conserve boson number, angular momentum and parity. We shall not explicitly write down all the numerous terms in such a hamiltonian, but rather discuss some of the associated transition operators, which are simpler to enumerate. Since we have bosons with both parities, we can construct transition operators both with positive and negative parity. Particularly important are the E1, E2 and E3 operators. The most general form of a one-boson E1 operator is

$$T^{(E1)} = \alpha_1(p^\dagger \times \tilde{s} + s^\dagger \times \tilde{p})^{(1)} + \beta_1(p^\dagger \times \tilde{d} + d^\dagger \times \tilde{p})^{(1)} + \gamma_1(f^\dagger \times \tilde{d} + d^\dagger \times \tilde{f})^{(1)}, \tag{2.3}$$

where we have used the standard notation for boson operators and their tensor products [11]). In a model without p-bosons only the last term in (2.3) survives. This term, by all accounts [10,12]) including our own extensive attempts to model octupole deformations without p-bosons, is unable to describe the observed properties of E1 transitions between low-lying collective states of opposite parity. The p-bosons thus appear to be a necessary ingredient in any successful phenomenological treatment of the data.

The E3 operator can be written as

$$T^{(E3)} = \alpha_3(f^\dagger \times \tilde{s} + s^\dagger \times \tilde{f})^{(3)} + \beta_3(p^\dagger \times \tilde{d} + d^\dagger \times \tilde{p})^{(3)} + \gamma_3(f^\dagger \times \tilde{d} + d^\dagger \times \tilde{f})^{(3)}. \tag{2.4}$$

E0, E2 and E4 operators are appropriately modified by the introduction of p- and f-bosons. For example, the E2 operator becomes

$$T^{(E2)} = \alpha_2(d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d})^{(2)} + \beta_2(d^\dagger \times \tilde{d})^{(2)} + \gamma_2(f^\dagger \times \tilde{p} + p^\dagger \times \tilde{f})^{(2)} + \delta_2(p^\dagger \times \tilde{p})^{(2)} + \eta_2(f^\dagger \times \tilde{f})^{(2)}. \tag{2.5}$$

## 3. Dynamic symmetries. The SU(3) limit

Since the algebra of U(16) is rather large, there are several possible dynamic symmetry chains that originating from it terminate in the rotation group O(3). For example, U(16) can be shown to contain all the chains of the s-d interacting boson model [13]), i.e. those going through U(5), SU(3) and O(6), in a non-trivial way. In addition, it has other chains which were not present in the U(6) model.

We begin by mentioning that, on the basis of microscopic calculations [8]), one expects a behavior of the single boson energies with boson number $N$ as shown schematically in fig. 3. Thus, there should be regions in which the p-boson has sufficiently high energy that it can be neglected. This corresponds to the separation

$$\mathrm{U}(16) \supset \mathrm{U}(13) \otimes \mathrm{U}(3) \tag{3.1}$$

and leads to a model of low-lying states in terms of U(13). As mentioned above, we have analyzed this model in detail and found it unable to describe octupole

deformations in a simple way [14]). If the energy of $f$-bosons is high enough, U(13) can be further separated into an s, d-part and an f-part leading to

$$\mathrm{U}(13) \supset \mathrm{U}(6) \otimes \mathrm{U}(7) \,. \tag{3.2}$$

This model, was discussed by Góźdź *et al.* [15]) and used with only one f-boson in refs. [5,12]). It describes octupole *vibrations* superimposed on quadrupole vibrations and/or rotations, and implies single boson energies $\varepsilon_{\mathrm{p}} \gg \varepsilon_{\mathrm{f}} \gg \varepsilon_{\mathrm{s}} \approx \varepsilon_{\mathrm{d}}$. Since it has been considered previously we shall not discuss it here further. Instead we consider the case in which all bosons have comparable energies $\varepsilon_{\mathrm{p}} \approx \varepsilon_{\mathrm{f}} \approx \varepsilon_{\mathrm{s}} \approx \varepsilon_{\mathrm{d}}$, leading to octupole deformations.

There are two dynamic symmetry chains in U(16) that produce rotational spectra (i.e. spectra composed of bands with energies increasing as $L(L+1)$). The first chain is

$$\begin{aligned} \mathrm{U}(16) &\supset \mathrm{U}(6) \otimes \mathrm{U}(10) \supset \mathrm{SU}(3)_a \otimes \mathrm{SU}(3)_b \\ &\supset \mathrm{SU}(3) \supset \mathrm{O}(3) \supset \mathrm{O}(2) \,, \qquad (\mathrm{I}) \,. \end{aligned} \tag{3.3}$$

This chain arises from the fact that the single s- and d-boson states transform as the (2, 0) representation of SU(3), while the p- and f-bosons transform as the (3, 0) representation. The labels $a$ and $b$ in (I) refer to the SU(3) algebras formed by s, d and p, f operators respectively. These algebras are generated by the operators

$$\begin{aligned} \hat{Q}_a &= (d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d})^{(2)} - \tfrac{1}{2}\sqrt{7}(d^\dagger \times \tilde{d})^{(2)} \,, \\ \hat{L}_a &= \sqrt{10}(d^\dagger \times \tilde{d})^{(1)} \,, \\ \hat{Q}_b &= (f^\dagger \times \tilde{p} + p^\dagger \times \tilde{f})^{(2)} + \tfrac{3}{2}\sqrt{\tfrac{3}{7}}(p^\dagger \times \tilde{p})^{(2)} + \sqrt{\tfrac{3}{2}}(f^\dagger \times \tilde{f})^{(2)} \,, \\ \hat{L}_b &= \sqrt{2}(p^\dagger \times \tilde{p})^{(1)} + 2\sqrt{7}(f^\dagger \times \tilde{f})^{(1)} \,. \end{aligned} \tag{3.4}$$

The total SU(3) algebra is generated by the sum of the operators of $\mathrm{SU}(3)_a$ and $\mathrm{SU}(3)_b$,

$$\begin{aligned} \hat{Q} &= \hat{Q}_a + \hat{Q}_b \,, \\ \hat{L} &= \hat{L}_a + \hat{L}_b \,. \end{aligned} \tag{3.5}$$

The energies of eigenstates for this symmetry can be obtained in the usual way by writing the hamiltonian in terms of Casimir invariants of the group chain (I),

$$\begin{aligned} H^{(\mathrm{I})} = {} & E_0 + \alpha_a C_1(\mathrm{U6}) + \beta_a C_2(\mathrm{U6}) + \alpha_b C_1(\mathrm{U10}) + \beta_b C_2(\mathrm{U10}) \\ & + \kappa_a C(\mathrm{SU3}_a) + \kappa_b C(\mathrm{SU3}_b) + \kappa C(\mathrm{SU3}) + \kappa' C(\mathrm{O3}) \,, \end{aligned} \tag{3.6}$$

where we have used the notation of ref. [11]). The eigenvalues are given by

$$\begin{aligned} & E(N, N_a, N_b, \lambda_a, \mu_a, \lambda_b, \mu_b, \omega, \lambda, \mu, K, L, M_L) \\ & \quad = E_0 + \alpha_a N_a + \beta_a N_a(N_a+5) + \alpha_b N_b + \beta_b N_b(N_b+9) + \kappa_a C(\lambda_a, \mu_a) \\ & \qquad + \kappa_b C(\lambda_b, \mu_b) + \kappa C(\lambda, \mu) + \kappa' L(L+1) \,, \end{aligned} \tag{3.7}$$

where [16])

$$C(\lambda, \mu) = \lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu \,. \tag{3.8}$$

The quantum numbers $\omega$, $K$ in (3.7) denote missing labels, while the other quantum numbers corresponds to the groups in the chain (3.3),

$$\left.\begin{array}{l} \mathrm{U}(16) \supset \mathrm{U}(6) \otimes \mathrm{U}(10) \supset \mathrm{SU}(3)_a \otimes \mathrm{SU}(3)_b \supset \\ \quad\downarrow \qquad \downarrow \qquad \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\ \quad N \qquad N_a \qquad N_b \quad (\lambda_a, \mu_a) \quad (\lambda_b, \mu_b) \\ \qquad \mathrm{SU}(3) \supset \mathrm{O}(3) \supset \mathrm{O}(2) \\ \qquad\quad \downarrow \qquad \downarrow \qquad \downarrow \\ \omega, (\lambda, \mu), K \quad L \qquad M_L \end{array}\right\} . \tag{3.9}$$

One can show, using the methods of ref. [11]), p. 151, that the total number of labels needed to classify uniquely the totally symmetric representations of U(16) is 16. The missing labels $\omega$ are thus four in number and they arise from the reduction of U(10) to $\mathrm{SU}(3)_b$. The actual decomposition of representations of U(6) and U(10) into those of $\mathrm{SU}(3)_a$ and $\mathrm{SU}(3)_b$ is described in ref. [16]). For $N_a$, $N_b \leq 4$ the results are summarized in table 1. In order to obtain the values of the quantum numbers $(\lambda, \mu)$ of the combined SU(3) group, one must take tensor products of representations of $\mathrm{SU}(3)_a$ and $\mathrm{SU}(3)_b$. For example,

$$(2, 0) \otimes (3, 0) = (5, 0) \oplus (3, 1) \oplus (1, 2) \,. \tag{3.10}$$

For appropriate values of the parameters, the lowest bands will have $N_a$ s, d bosons and $N_b$ p, f bosons. Note that, since $N = N_a + N_b$, two of the terms in (3.7) can be

TABLE 1

Decomposition of the representations of U(6) and U(10) into those of $\mathrm{SU}(3)_a$ and $\mathrm{SU}(3)_b$

| U(6) | $\mathrm{SU}(3)_a$ |
|---|---|
| [1] | (2, 0) |
| [2] | (4, 0), (0, 2) |
| [3] | (6, 0) (2, 2) (0, 0) |
| [4] | (8, 0) (4, 2) (0, 4) (2, 0) |
| **U(10)** | **$\mathrm{SU}(3)_b$** |
| [1] | (3, 0) |
| [2] | (6, 0) (2, 2) |
| [3] | (9, 0) (5, 2) (3, 3) (3, 0) (0, 3) |
| [4] | (12, 0) (8, 2) (6, 3) (6, 0) (3, 3) (4, 4) (4, 1) (0, 6) (2, 2) (0, 0) |

eliminated and included in $E_0$, yielding

$$E(N, N_a, N_b, \lambda_a, \mu_a, \lambda_b, \mu_b, \omega, \lambda, \mu, K, L, M_L) = E_0' + \alpha' N_b + \beta' N_b^2 + \kappa_a C(\lambda_a, \mu_a) + \kappa_b C(\lambda_b, \mu_b) + \kappa C(\lambda, \mu) + \kappa' L(L+1) . \tag{3.11}$$

If $N_b \neq 0$ in the lowest bands, eq. (3.11) describes a system with octupole *deformations* (i.e. the ground state is a condensate that includes f-bosons). The corresponding spectrum is shown in fig. 4.

Eq. (3.11) does not however describe the spectrum of a *rigid body* with octupole deformations. One property of this latter spectrum is the occurrence of *parity doublets*, as observed for example in molecules ($NH_3$ molecule). Parity doublets can be obtained in this chain only through a rather artificial procedure: the inclusion of

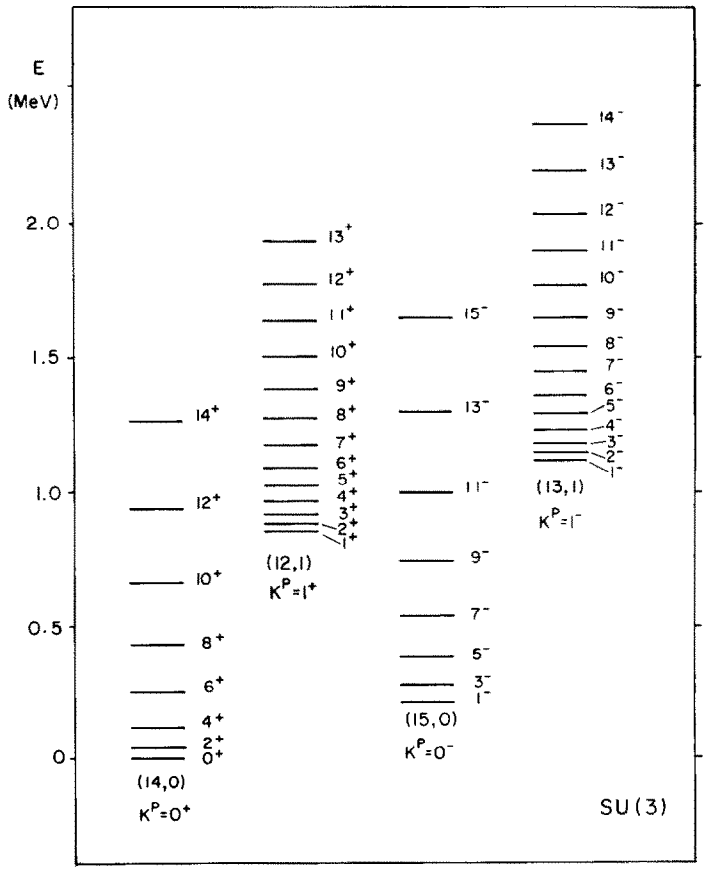


Fig. 4. Schematic representation of the spectrum of a nucleus in the SU(3) limit of the U(16) model with $N = 6$. The lowest positive parity bands belonging to the configuration $N_a = 4$, $N_b = 2$ and negative-parity bands belonging to $N_a = 3$, $N_b = 3$ are shown. The excitation energies are computed using eq. (3.11) with $\alpha' = -693.4$ keV, $\beta' = 346.7$ keV, $\kappa_a = \kappa_b = 0$, $\kappa = -20$ keV, $\kappa' = 6$ keV.

signature-dependent interactions, i.e. interactions that depend on parity, $(-)^{Nb}$. Including the appropriate signature-dependent forces in $H^{(1)}$ leads to the energy formula [17])

$$E(N, N_a, N_b, \lambda_a, \mu_a, \lambda_b, \mu_b, \omega, \lambda, \mu, K, L, M_L)$$
$$= E_0' + \alpha' N_b + \beta' \left[1 - \frac{(-)^{N+N_b}}{N_b}\right] N_b^2 + \kappa_a \left[1 + \frac{(-)^{N+N_b}}{N_a}\right] C(\lambda_a, \mu_a)$$
$$+ \kappa_b \left[1 - \frac{(-)^{N+N_b}}{N_b}\right] C(\lambda_b, \mu_b) + \kappa \left[1 - \frac{(-)^{N+N_b}}{2N_b + 2N_a}\right] C(\lambda, \mu) + \kappa'(L(L+1)\,. \tag{3.12}$$

The corresponding spectrum in the limit of large $N_a$ and $N_b$ is shown in fig. 5.

We will have more to say about parity doubling in SU(3) in the next section. Here we digress briefly in order to clarify the complex vibrational structure of fig. 5. To this end, we consider a model with only s, p and f bosons (no quadrupole

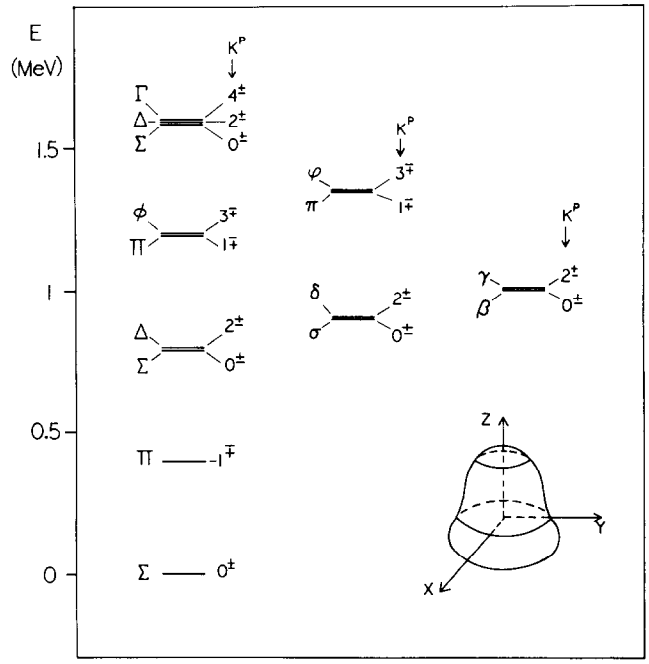


Fig. 5. Schematic representation of the vibrational spectrum of a nucleus with both quadrupole and octupole deformations. The spectrum is calculated from eq. (3.12) in the large-$N$ limit with appropriately selected parameters. There are three types of vibrations, with scales set by $\kappa$, $\kappa_a$ and $\kappa_b$. The first of these is labelled with molecular spectroscopic symbols ($\Sigma$, $\Pi$, $\Delta$, ...) in addition to the parity and value of $K$. The second type if labeled as in fig. 6 and the third type according to the usual nuclear quadrupole notation ($\beta$ and $\gamma$). Other sets of vibrational bands, corresponding to different values of $N_a$ and $N_b$, have been pushed higher up in the spectrum and are not shown in the figure. On top of each vibrational state shown here, there is built a rotational band as in fig. 4.

degree of freedom). The corresponding algebraic structure is that of U(11). The same type of analysis that led to eq. (3.12) can be applied to this model, since the s-boson transforms as the (0, 0) representation of SU(3). Paralleling the steps described above, one then obtains the energy formula

$$E(N, N_b, \lambda_b, \mu_b, \omega, K, L, M_L)$$

$$= E_0 + \alpha' N_b + \beta' \left[1 - \frac{(-)^{N+N_b}}{N_b}\right] N_b^2 + \kappa_b \left[1 - \frac{(-)^{N+N_b}}{N_b}\right] C(\lambda_b, \mu_b) + \kappa' L(L+1) . \tag{3.13}$$

The low-lying part of the spectrum of eq. (3.13) is shown in fig. 6. Particularly interesting is the structure of the lowest (parity doubled) vibrations; they have quantum numbers $K^P = 0^{\pm}, 2^{\pm}, 1^{\pm}, 3^{\pm}$ and are denoted here by $\sigma$, $\delta$, $\pi$ and $\phi$, respectively. These four vibrational bands are the octupole analogs of the two vibrational bands $\beta$ and $\gamma$ of a quadrupole deformed body. In fig. 5, which represents the spectrum associated with a body having both octupole and quadrupole deformations, both the $\sigma$, $\delta$, $\pi$, $\phi$ and $\beta$, $\gamma$ types of oscillations are present, along with modes

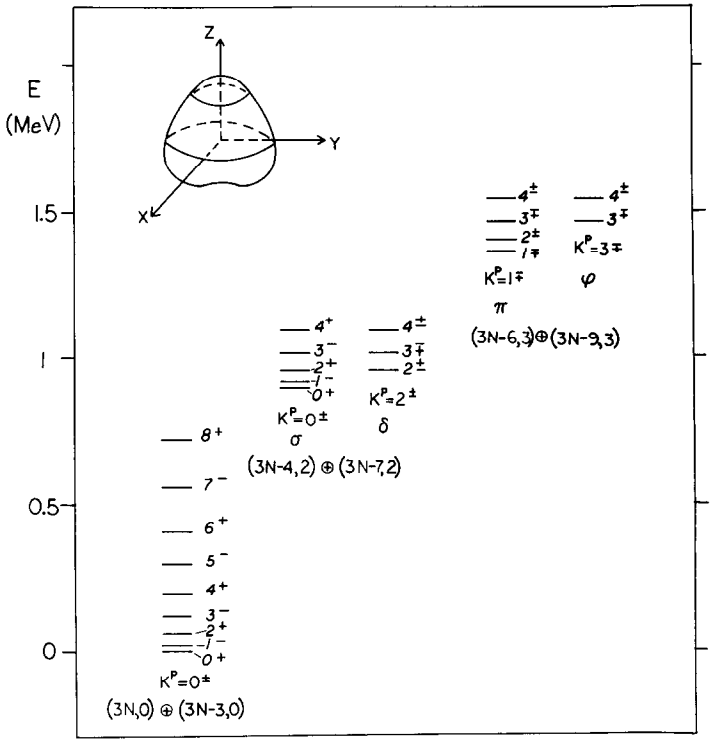


Fig. 6. Schematic representation of the spectrum of a nucleus with purely octupole-dipole deformations. The spectrum is calculated using eq. (3.13) with $\alpha' = \beta' = 0$ in the large-$N$ limit. The rotational constants are chosen to be $\kappa_b = (-50/N)$ keV and $\kappa' = 10$ keV.

we have labelled $\Sigma, \Pi, \Delta, \ldots$ that correspond to the motion of the deformed quadrupole and octupole shapes with respect to one another.

We turn now to a discussion of electromagnetic transition rates. For the most general forms of the operators, given in eqs. (2.3)–(2.5), these are somewhat difficult to compute. However, if the operators have simple tensorial properties under the groups considered, they can be calculated simply by using a generalized Wigner–Eckart theorem [18]). For example, if the E2 operator is

$$T^{(\mathrm{E2})} = \alpha_2 \hat{Q}_a + \gamma_2 \hat{Q}_b \,, \tag{3.14}$$

its matrix elements in the SU(3) chain can be calculated easily. Similarly, one can calculate the matrix elements of the E1 operator constructed with s, d and p, f bosons if it transforms as a tensor of rank (1, 0) under SU(3). Consider, for example, the case in which the lowest positive parity band has quantum numbers $(\lambda_a, 0)$, $(\lambda_b, 0)$, $(\lambda = \lambda_a + \lambda_b, 0)$ and the lowest negative parity band has $(\lambda_a + 2, 0)$, $(\lambda_b - 3, 0)$, $(\lambda = \lambda_a + \lambda_b - 1, 0)$. Then one obtains

$$B(\mathrm{E1}; (\lambda, 0), L \to (\lambda - 1, 0), L-1) = g(\lambda_a, \lambda_b, \lambda) \frac{\lambda + L + 1}{2L+1} L \,,$$

$$B(\mathrm{E1}; (\lambda - 1, 0), L \to (\lambda, 0), L-1) = g(\lambda_a, \lambda_b, \lambda) \frac{\lambda - L}{2L+1} L \,. \tag{3.15}$$

The coefficient $g(\lambda_a, \lambda_b, \lambda)$ does not depend on $L$. In the limit of large $\lambda$, these values go over into the values given by the Alaga rule

$$B(\mathrm{E1}; L_\mathrm{i} \to L_\mathrm{f}) = g \langle L_\mathrm{i}, 0, 1, 0 | L_\mathrm{f}, 0 \rangle^2 \,, \tag{3.16}$$

corresponding to rigid deformations. Similar results apply to other operators and are given in ref. [14]).

## 4. Dynamic symmetries. The O(4) limit

Within U(16), there is another group chain that produces rotational spectra. Although we believe, for reasons discussed below, that this chain is not appropriate for describing octupole deformations in nuclei, we nonetheless discuss it here in order to point out differences and similarities with chain I. Furthermore, this chain can be used in other fields of physics, i.e. molecular physics. Its existence is due to the fact that s, d, p and f bosons transform as the (3, 0) representation of O(4). Thus, the algebra of O(4) is contained in U(16) and can be used to generate a dynamic symmetry.

A reduction of representations of U(16) to representations of O(4) and the construction of the intermediate subalgebras is rather complex. A convenient way to perform it is by making a transformation on the boson operators $b_{l,\mu}$ $(l = 0, 1, 2, 3)$, introducing the operators

$$a^\dagger_{m,m'} = \Sigma_l \langle \tfrac{3}{2}, m, \tfrac{3}{2}, m' | l, m + m' \rangle b^\dagger_{l,m+m'} \,. \tag{4.1}$$

The operators $a^\dagger$ carry two indices $m$ and $m'$ which can take the values $\pm\frac{3}{2}, \pm\frac{1}{2}$. The action of the parity operator, $P$, on these objects is given by

$$Pa^\dagger_{m,m'}P^{-1} = -a^\dagger_{m',m}\,. \tag{4.2}$$

This follows from the relation

$$Pb^\dagger_{l,\mu}P^{-1} = (-)^l b^\dagger_{l,-\mu} \tag{4.3}$$

and the symmetry properties of Clebsch-Gordan coefficients under permutation of indices.

The generators of U(16), composed of all the bilinear products $b^\dagger_{l,\mu}b_{l',\mu'}$, can be rewritten in terms of the bilinear products $a^\dagger_{m,m'}a_{n,n'}$. A summation over the first (or second) index of the creation and annihilation operators allows one to construct from the operators $a^\dagger_{m,m'}a_{n,n'}$ two commuting U(4) algebras and thus to break U(16) into U(4)⊗U(4). In order to distinguish the two U(4)'s, we will attach labels $a$ and $b$ to the appropriate groups. Each U(4) can then be further broken down by using the decomposition [19])

$$\mathrm{U}(4) \supset \mathrm{Sp}(4,\mathrm{C}) \approx \mathrm{O}(5) \supset \mathrm{SU}(2) \approx \mathrm{O}(3)\,. \tag{4.4}$$

This procedure leads to the chain

$$\begin{aligned}\mathrm{U}(16) &\supset \mathrm{U}(4)_a \otimes \mathrm{U}(4)_b \supset \mathrm{Sp}(4)_a \otimes \mathrm{Sp}(4)_b \supset \mathrm{SU}(2)_a \otimes \mathrm{SU}(2)_b \\ &\approx \mathrm{O}(4) \supset \mathrm{O}(3) \supset \mathrm{O}(2)\,, \qquad \text{(II)}\end{aligned} \tag{4.5}$$

where we have deleted the letter C from Sp(4, C) following common practice. The symbol ≈ in (4.4) and (4.5) denotes isomorphic algebras. The O(4) algebra contained in (4.5) is generated by *six* operators, divided into a vector and a pseudovector. These six operators can be written in terms of the original boson operators $b_{l,\mu}$ by performing the transformation (4.1) backwards. They correspond physically to the angular momentum operator $\hat{L}$,

$$\hat{L} = \sqrt{2}(p^\dagger \times \tilde{p})^{(1)} + \sqrt{10}(d^\dagger \times \tilde{d})^{(1)} + 2\sqrt{7}(f^\dagger \times \tilde{f})^{(1)} \tag{4.6}$$

and the dipole operator $\hat{D}$,

$$\hat{D} = \sqrt{5}(p^\dagger \times \tilde{s} + s^\dagger \times \tilde{p})^{(1)} - 2\sqrt{2}(d^\dagger \times \tilde{p} + p^\dagger \times \tilde{d})^{(1)} + \sqrt{7}(f^\dagger \times \tilde{d} + d^\dagger \times \tilde{f})^{(1)} \tag{4.7}$$

The basis states for the group chain (II) are labelled as follows

$$\left|\begin{array}{ccccccc} \mathrm{U}(16) \supset & \mathrm{U}(4)_a & \otimes & \mathrm{U}(4)_b & \supset & \mathrm{Sp}(4)_a & \otimes\ \mathrm{Sp}(4)_b \\ \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \mathrm{N} & (n_1, n_2, n_3, n_4) & & (n_1, n_2, n_3, n_4) & & (n'_{1a}, n'_{2a}) & (n'_{1b}, n'_{2b}) \\ \multicolumn{7}{l}{\supset \mathrm{SU}(2)_a \times \mathrm{SU}(2)_b \supset \mathrm{O}(3) \supset \mathrm{O}(2)} \\ \downarrow & \downarrow & \downarrow & \downarrow & & & \\ \nu & j_a & j_b & L & M_L & & \end{array}\right\rangle, \tag{4.8}$$

where we have indicated by $\nu$ any missing labels. The fact that we are dealing with totally symmetric representations of U(16) requires that the quantum numbers $n_1, n_2, n_3, n_4$ be identical for $U(4)_a$ and $U(4)_b$. A method similar to that discussed in sect. 3, eq. (3.9), shows that, in this case, the missing labels are three in number. In eq. (4.8) the representations of O(4) are labelled by the $SU(2)_a \otimes SU(2)_b$ labels $(j_a, j_b)$. It is straightforward to convert these labels to the usual O(4) labels $(\omega, \omega')$ by using

$$\omega = j_a + j_b,$$
$$\omega' = |j_a - j_b| . \tag{4.9}$$

The representations of O(4) contained in a given totally symmetric representation of U(16), $[N]$, are obtained by standard rules after noting that the fundamental representation [1] of U(16) corresponds to the representation $(1,0,0,0)\otimes(1,0,0,0)$ of $U(4)_a \otimes U(4)_b$. (One needs to know as well the decomposition of representations of U(4) into those of SU(2). This step is discussed in ref. [19]).) The results are presented for $N \leq 3$ in table 2. For larger $N$'s they can be obtained as discussed above. An important result is that whenever a representation $(j_a, j_b)$ with $j_a \neq j_b$ appears, so does the representation $(j_b, j_a)$. The representation $(j_a, j_b)$ contains angular momenta

$$L = |j_a - j_b|, |j_a - j_b| + 1, \ldots, |j_a + j_b| . \tag{4.10}$$

TABLE 2

Decomposition of the representations of U(16) into those of O(4)

| U(16) | $U(4)_a \otimes U(4)_b$ | $Sp(4)_a \otimes Sp(4)_b$ | $SU(2)_a \otimes SU(2)_b \approx O(4)$ | |
|---|---|---|---|---|
| $[N]$ | $(n_1, n_2, n_3, n_4)$ $\otimes$ $(n_1, n_2, n_3, n_4)$ | $(n'_{1a}, n'_{2a})$ $\otimes$ $(n'_{1b}, n'_{2b})$ | $j_a \otimes j_b$ | $(\omega, \omega')$ |
| [1] | $(1,0,0,0)\otimes(1,0,0,0)$ | $(1,0)\otimes(1,0)$ | $\frac{3}{2}\otimes\frac{3}{2}$ | (3, 0) |
| [2] | $(2,0,0,0,0)\otimes(2,0,0,0)$ | $(2,0)\otimes(2,0)$ | 3; $1\otimes 3$; 1 | (6, 0)(4, 2) (2, 0) |
| | $(1,1,0,0)\otimes(1,1,0,0)$ | $\binom{(1,1)}{(0,0)}\otimes\binom{(1,1)}{(0,0)}$ | $\binom{2}{0}\otimes\binom{2}{0}$ | (4, 0)(2, 2) (0, 0) |
| [3] | $(3,0,0,0)\otimes(3,0,0,0)$ | $(3,0)\otimes(3,0)$ | $\frac{9}{2}; \frac{5}{2}; \frac{3}{2}$ $\otimes$ $\frac{9}{2}; \frac{5}{2}; \frac{3}{2}$ | (9, 0) (7, 2) (6, 3) (5, 0) (4, 1)(3, 0) |
| | $(2,1,0,0)\otimes(2,1,0,0)$ | $\binom{(2,1)}{(1,0)}\otimes\binom{(2,1)}{(1,0)}$ | $\binom{\frac{7}{2}; \frac{5}{2}; \frac{1}{2}}{\frac{3}{2}}$ $\otimes$ $\binom{\frac{7}{2}; \frac{5}{2}; \frac{1}{2}}{\frac{3}{2}}$ | (7, 0) (6, 1) (5, 2) (5, 0) (4, 3) (4, 1) (3, 2) (3, 0) (2, 1) (1, 0) |
| | $(1,1,1,0)\otimes(1,1,1,0)$ | $(1,0)\otimes(1,0)$ | $\frac{3}{2}\otimes\frac{3}{2}$ | (3, 0) |

From the above and the action of the parity operator given in (4.2), one can conclude that all representations with $|j_a - j_b| \neq 0$ will be parity doubled. Explicit construction of the representations with $j_a = j_b$, i.e. $\omega' = 0$, shows that they alternate in parity. For example, the representation (6, 0) in table 2 contains $0^+, 1^-, 2^+, 3^-, 4^+, 5^-, 6^+$.

We now consider explicitly the dynamic symmetry associated with the group chain II. Although in principle we could consider Hamiltonians containing Casimir invariants of all the groups appearing in (4.5), the intermediate groups contribute little to the physics and we shall thus restrict our attention to the hamiltonian

$$H^{(II)} = E_0 - A\hat{D}\cdot\hat{D} + B\hat{L}\cdot\hat{L} = E_0 - AC(O4) + (B+A)C(O3)\,, \qquad (4.11)$$

where we have used the fact that the quadratic Casimir invariant of O(4) is given by

$$C(O4) = \hat{D}\cdot\hat{D} + \hat{L}\cdot\hat{L}\,. \qquad (4.12)$$

The energy eigenvalues of (4.11) are given by

$$E(N, n_1, n_2, n_3, n_4, n'_{1a}, n'_{2a}, n'_{1b}, n'_{2b}, \nu, j_a, j_b, L, M_L)$$
$$E_0 - 2A[j_a(j_a+1) + j_b(j_b+1)] + (B+A)L(L+1)\,. \qquad (4.13)$$

The term $j_a(j_a+1) + j_b(j_b+1)$ can be rewritten in the form $\frac{1}{2}[\omega(\omega+2) + \omega'^2]$. The spectrum associated with (4.13) is shown in fig. 7.

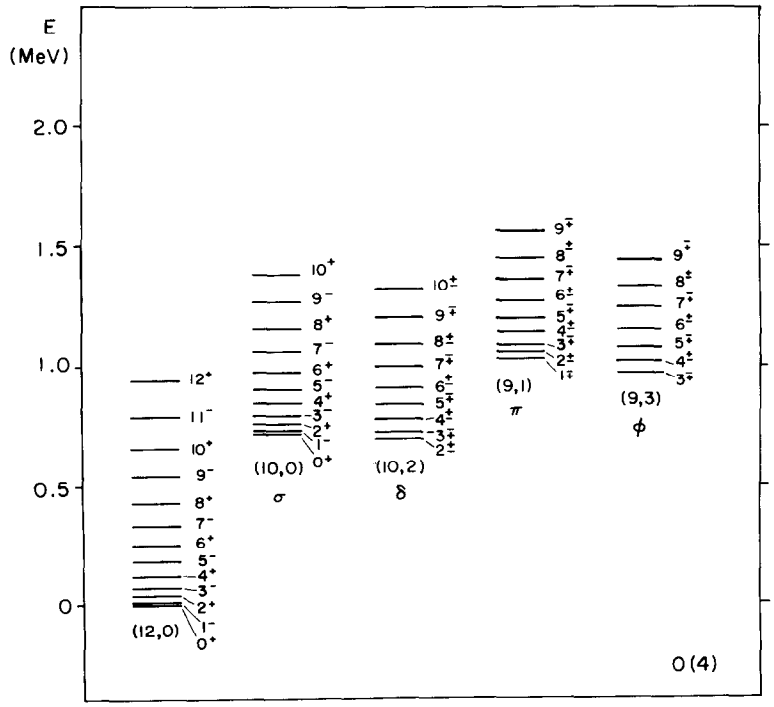


Fig. 7. Schematic representation of the spectrum in the O(4) limit of the U(16) model and $N = 4$. The excitation energies are computed using eq. (4.13) with $A = 15$ keV, $B = -9$ keV. Only the low-lying portion of the spectrum is shown.

Electromagnetic transition rates can also be computed using standard techniques. To this end, it is convenient to rewrite the transition operators in terms of tensor operators that have well-defined properties with respect to the groups appearing in eq. (4.8). The dependence of the matrix elements on the angular momenta of the states can then be obtained easily, following the discussion of ref. [18]), p. 240. If we denote by $T^{\gamma,(a\times b),\lambda}$ a tensor operator of multipolarity $\lambda$, that transforms under $SU(2)_a \otimes SU(2)_b$ as the representation $(a\times b)$ and under the other groups as the representations globally denoted by $\gamma$, we can write the following expression for the matrix elements between states that transform as $|\alpha, (j_a \times j_b), L\rangle$:

$$\langle \alpha, (j_a \times j_b), L \| T^{\gamma,(a\times b),\lambda} \| \alpha', (j'_a \times j'_b), L' \rangle$$
$$= \sqrt{(2L+1)(2L'+1)(2\lambda+1)} \begin{Bmatrix} j_a & j_b & L \\ j'_a & j'_b & L' \\ a & b & \lambda \end{Bmatrix} \langle \alpha, (j_a \times j_b) \| T^{\gamma,(a\times b)} \| \alpha', (j'_a \times j'_b) \rangle , \tag{4.14}$$

where the last matrix element is "doubly reduced". The entire angular momentum dependence is contained in the first two terms in the rhs of eq. (4.14). From this expression one can extract information on the behavior of the matrix elements. For example, since

$$\begin{Bmatrix} j & j & L \\ j & j & L' \\ a & b & \lambda \end{Bmatrix} \xrightarrow[j\to\infty]{} (-)^{L+b} \frac{\langle L'0, \lambda 0 | L0 \rangle \langle a0, \lambda 0 | b0 \rangle}{(2j+1)\sqrt{(2L+1)(2L'+1)}} , \tag{4.15}$$

one obtains, for states with $j_a = j'_a = \frac{3}{2}N$, $j_b = j'_b = \frac{3}{2}N$ (ground state band),

$$B(E\lambda; L' \to L) \xrightarrow[N\to\infty]{} f(a, b, \lambda, N) \langle L'0, \lambda 0 | L0 \rangle^2 , \tag{4.16}$$

i.e. the Alaga rule. (Here $f$ is an $L, L'$ independent factor.) Explicit expressions, valid for any $N$, can be obtained by evaluating the last term in the rhs of eq. (4.14). The evaluation is particularly simple for E1 transitions and in the case in which the transition operator is proportional to the dipole operator $\hat{D}$ of eq. (4.7). Since $\hat{D}$ is a generator of $O(4) \approx SU(2)_a \otimes SU(2)_b$, excited bands will not be connected to the ground band, and only intraband transitions will be allowed. Using a technique similar to that discussed in ref. [18]), p. 304 one obtains

$$B[E1; (\tfrac{3}{2}N, \tfrac{3}{2}N), L \to (\tfrac{3}{2}N, \tfrac{3}{2}N), L-1] = \theta^2 \frac{(3N+L-1)(3N-L+1)L}{2L+1} , \tag{4.17}$$

where $\theta$ is the proportionality constant between $T^{(E1)}$ and $\hat{D}$.

For all the reasons mentioned above, and in particular because of the natural occurrence of parity doubling, O(4) appears to provide the most faithful representation of asymmetric shape deformations. However, this symmetry implies strong

*dipole* interactions. While these are the dominant forces in molecules, in nuclei the strongest interactions are of pairing and quadrupole type. We are therefore led to the conclusion that the O(4) chain is not as relevant in nuclei as the SU(3) chain and that parity doubling, to the extent that it occurs, remains an accidental fact. Although we have obtained this result from an analysis of boson dynamic symmetries, we suspect that this statement is true of any bosonic hamiltonian not incorporating dipole interactions. It is an open question as well to what extent this statement is true of other realizations of octupole deformations, e.g. shape variables. In this respect, it would be interesting to produce Bohr-like hamiltonians with continuous, differentiable potential functions (i.e. not parity-dependent or discontinuous potentials) which yield rotational spectra with parity doubling.

## 5. Classical limit

We have seen, through examination of spectra and transitions, that the dynamical symmetries discussed above yield properties similar to those of rotating deformed shapes. To make the relation more explicit, it is helpful to study the classical (or geometric) limit of our boson hamiltonians. In the shape variable description one parametrizes the nuclear surface as

$$R = R_0(1+\Sigma_\mu \alpha_{1\mu} Y_{1\mu}(\theta,\phi)+\Sigma_\mu \alpha_{2\mu} Y_{2\mu}(\theta,\phi)+\Sigma_\mu \alpha_{3\mu} Y_{3\mu}(\theta,\phi))\,. \tag{5.1}$$

Geometric properties of boson models can be studied by considering coherent states of the form

$$|[N],\alpha\rangle = \frac{1}{N!}(\Sigma_l \alpha_l \cdot b_l^\dagger)^N|0\rangle\,, \tag{5.2}$$

where the dot denotes a scalar product. The $c$-numbers $\alpha_{l,\mu}$ in (5.2) are related to the parameters $\alpha_{l'\mu}$ in (5.1), by a simple renormalization (see, for example, p. 105 of ref. [11])). The total energy surface corresponding to a given hamiltonian $H$ is obtained from

$$E(\alpha) = \frac{\langle[N],\alpha|H|[N],\alpha\rangle}{\langle[N],\alpha|[N],\alpha\rangle}\,. \tag{5.3}$$

In the case of pure quadrupole deformations, the study of the properties of $E(\alpha)$ is relatively simple. In the presence of all the degrees of freedom considered here ($l=0,1,2,3$) it is rather complex. We have therefore analyzed only the two special cases discussed in the previous sections.

The analysis of the dynamic symmetry (II) is straightforward. The O(4) hamiltonian (4.11) yields in the large $N$ limit an energy surface with a well defined minimum at

$$\alpha_{10}^{eq} = 3/\sqrt{5}, \qquad \alpha_{20}^{eq} = 1, \qquad \alpha_{30}^{eq} = 1/\sqrt{5}, \qquad \alpha_{l\mu}^{eq} = 0 \quad (\mu \neq 0)\,. \tag{5.4}$$

The system has a simultaneous dipole, quadrupole and octupole deformation. The values of the deformations are all related.

The analysis of the dynamic symmetry (I) is more difficult. Because the hamiltonian separately conserves $N_a$ and $N_b$ in (3.9), the coherent state (5.2) will not yield a stable minimum in $E(\alpha)$. A more suitable coherent state has the form

$$(s^\dagger + \alpha_2 \cdot d^\dagger)^{N_a}(\alpha_1 \cdot p^\dagger + \alpha_3 \cdot f^\dagger)^{N_b}|0\rangle\,, \tag{5.5}$$

i.e. a state with fixed $N_a$ and $N_b$. The SU(3) hamiltonian (3.5) yields an energy surface with a stable minimum at

$$\alpha_{20}^{\mathrm{eq}} = \sqrt{2}\,, \qquad (\alpha_{30}/\alpha_{10})^{\mathrm{eq}} = -\sqrt{\tfrac{2}{3}}\,,$$
$$\alpha_{10}^{\mathrm{eq}} = \text{arbitrary}\,, \qquad \alpha_{l\mu}^{\mathrm{eq}} = 0 \qquad (\mu \neq 0)\,. \tag{5.6}$$

(The arbitrariness in $\alpha_{10}^{\mathrm{eq}}$ reflects freedom in the way the state (5.5) is normalized.) This case also represents a system with simultaneous dipole, quadrupole and octupole deformation, but where the quadrupole deformation is independent from the dipole-octupole deformation.

Since in both cases, the minima occur for values of $\alpha_{l,\mu} \neq 0$ only when $\mu = 0$, both symmetries correspond to *axial* deformations in which the axes of the various deformations are all aligned. An example of such a shape is shown in the insert of fig. 5. The study of triaxial deformations and/or cases in which the axes are not aligned is very complex. This, together with the geometric interpretation of the normal modes of the system, denoted by $\Pi, \Sigma, \Delta, \ldots$ in fig. 5, could be done using the techniques developed recently by Leviatan [20]).

## 6. Numerical studies

A major problem in studying octupole degrees of freedom in nuclei is the lack of experimental data. This situation is very different from that encountered in the study of quadrupole degrees of freedom. Four possible interpretations of collective low-lying negative-parity states in nuclei have been suggested: (i) octupole vibrations [4,21,22]); (ii) octupole deformations [6]); (iii) dipole vibrations [23]) and (iv) dipole deformations [23]). We have included here (iii) and (iv) for completeness although we shall not discuss them in the context of the present work. It is well known from the study of the quadrupole case that what distinguishes between a deformation and a vibration is the structure of the side bands. For example, a body with quadrupole deformation has excited $\beta$ and $\gamma$ bands while a spherical body has two and three phonon multiplets. Unfortunately, only the lowest negative-parity band is usually known. On the basis of the energies of this band it is not possible to distinguish between deformations and vibrations. The only measured quantity that appears to be somewhat sensitive to different physical interpretations is the ratio

$$R = \frac{B(\mathrm{E1}; J \to J-1)}{B(\mathrm{E2}; J \to J-2)}\,, \qquad J^P = 2^+, 3^-, 4^+, 5^-, \ldots\,, \tag{6.1}$$

between states in the lowest positive- and negative-parity bands. We shall therefore concentrate our discussion on this quantity.

We first remark that the behavior of $R$ with $J$ does not support the interpretation of the negative parity levels as simple f-boson excitations (octupole vibrations). In a model without p-bosons, the one-body E1 operator necessarily has the form

$$T^{(E1)} = \alpha_1 (d^\dagger \times \tilde{f} + f^\dagger \times \tilde{d})^{(1)} . \tag{6.2}$$

This operator, in an octupole vibration picture, produces E1 transitions that are much larger in the $J$-odd to $J$-even direction than they are in the opposite direction, contrary to the observed data which indicate relative equality of the two types of transitions. This deficiency was remedied in ref. [12]) through the addition of higher order terms to the operator (6.2). However, even with these terms, the final fit relied on a delicate balancing of the one- and two-body terms in the operator. It seems to be necessary to introduce p-bosons to obtain the smoothness in $R$ observed in experiment. Even within U(16), there are two possible schemes: (i) octupole-dipole vibrations; (ii) octupole-dipole deformations. We have analyzed the available experimental data in the Ra region in terms of both pictures and present some of the results below. In doing this analysis one may attempt to use the dynamic symmetries discussed above, i.e.

$$\mathrm{U}(16) \supset \mathrm{U}(6) \otimes \mathrm{U}(7) \otimes \mathrm{U}(3) \tag{6.3}$$

for vibrational spectra and

$$\mathrm{U}(16) \begin{cases} \nearrow \mathrm{SU}(3)_a \otimes \mathrm{SU}(3)_b \supset \mathrm{SU}(3) & \text{(I)} \\ \searrow \mathrm{U}(4)_a \otimes \mathrm{U}(4)_b \supset \mathrm{O}(4) & \text{(II)} \end{cases} \tag{6.4}$$

for rotational spectra. Unfortunately, this cannot be done for several reasons:

(i) the radium isotopes are in a transitional region with respect to the quadrupole (U(5) spherical → SU(3) deformed transition);

(ii) none of the typical features of the chains (I) and (II) appear to be present in the data.

We are thus led to the conclusion that both the quadrupole and the octupole degrees of freedom are in a soft transitional region. Numerical calculations are then needed in order to study the situation. To this end, one of us (JE) wrote a computer program [24]) that diagonalizes the U(16) hamiltonian for $N \leq 10$. The complexity of the problem clearly increases considerably with increasing N. Rohozinski [7]) has listed the states of $f^N$ configuration for $N \leq 8$. As one can see from his table, the number increases rapidly. In the diagonalization one needs the coefficients of fractional parentage (c.f.p.) for f-bosons. These were generated with the procedure discussed in ref. [25]). Because of the limitation $N \leq 10$, our calculations could not be carried out in Ra beyond $A = 228$. Our results are as follows:

(i) *Vibrations.* We have used here the hamiltonian

$$H = \varepsilon_{\mathrm{d}} \hat{n}_{\mathrm{d}} + \kappa \hat{Q}_{\mathrm{sd}} \cdot \hat{Q}_{\mathrm{sd}} + \kappa' \hat{L}_{\mathrm{d}} \cdot \hat{L}_{\mathrm{d}} + \varepsilon_{\mathrm{p}} \hat{n}_{\mathrm{p}} + \varepsilon_{\mathrm{f}} \hat{n}_{\mathrm{f}} + \eta (\hat{n}_{\mathrm{f}} + \hat{n}_{\mathrm{p}})^2 + \kappa'' \hat{Q}_{\mathrm{a}} \cdot \hat{Q}_{\mathrm{b}} + \kappa''' \hat{L}_{\mathrm{a}} \cdot \hat{L}_{\mathrm{b}} , \tag{6.5}$$

where $\hat{n}$ denotes number operators, $\hat{Q}_a$, $\hat{Q}_b$, $\hat{L}_a$ and $\hat{L}_b$ are the operators in (3.4) and $\hat{Q}_{sd}$ is the usual quadrupole operator of s, d bosons [11]),

$$\hat{Q}_{sd} = (d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d})^{(2)} + \chi (d^\dagger \times \tilde{d})^{(2)}, \tag{6.6}$$

with $\chi = -0.75 =$ fixed. This hamiltonian is suggested by microscopic considerations under the assumption that the dominant interactions are of pairing and quadrupole type. The parameter values are listed in table 3 and the corresponding fit to the energies is shown in fig. 8. Since we want here to analyze a vibration-like situation, we have restricted the calculations to $n_p$, $n_f \leq 1$. The ground-state band thus contains

TABLE 3

Parameters (MeV) of the 1pf fit to the radium isotopes

| Isotope | $\epsilon_d$ | $\kappa$ | $\kappa'$ | $\varepsilon_f$ | $\varepsilon_p$ | $\eta$ | $\kappa''$ | $\kappa'''$ |
|---|---|---|---|---|---|---|---|---|
| $^{220}$Ra | 0.43 | −0.035 | −0.0128 | 0.732 | 1.152 | 0.050 | −0.1410 | 0.0174 |
| $^{222}$Ra | 0.35 | −0.035 | −0.0097 | 0.742 | 1.242 | 0.050 | −0.1396 | 0.0142 |
| $^{224}$Ra | 0.35 | −0.035 | −0.0097 | 0.742 | 1.242 | 0.050 | −0.1271 | 0.0142 |
| $^{226}$Ra | 0.35 | −0.035 | −0.0099 | 0.742 | 1.242 | 0.050 | −0.1327 | 0.0142 |

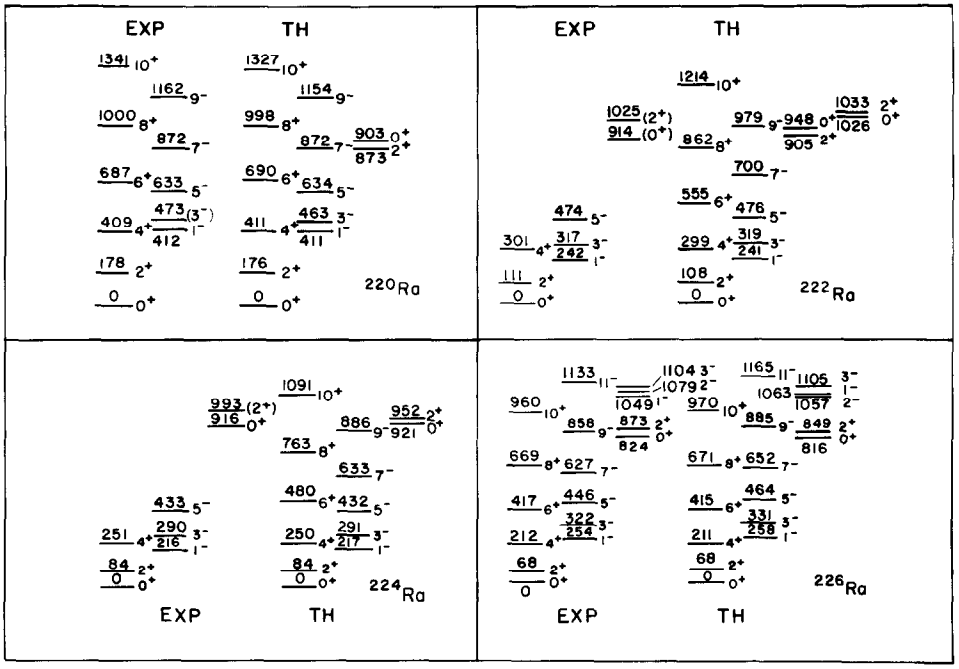


Fig. 8. Measured spectra in radium isotopes and U(16) theoretical fits in the vibrational limit (one pf boson in the negative-parity band). The hamiltonian is given in eq. (6.5). Parameter values are in table 3.

no negative parity bosons, while the first negative-parity band contains one negative-parity boson.

Using the wave functions obtained from the numerical diagonalization, the E1 operator of eq. (2.3) and the E2 operator

$$T^{(E2)} = \alpha_2' \hat{Q}_{sd} + \beta_2' (f^\dagger \times \tilde{f})^{(2)}, \tag{6.7}$$

we then calculated the ratio $R$. The comparison with experiment in $^{220}$Ra is shown in fig. 9. Although the observed values can be reproduced fairly well, the fit requires a fine tuning of parameters similar to that discussed above. The reason is that, in this vibrational picture, each of the three terms in (2.3) produces transitions primarily in one direction. This fine tuning is a somewhat undesirable feature of our fit.

(ii) *Deformations.*

Because of the complexity of the problem we have not analyzed all possible deformation schemes. One way to produce octupole deformations is as discussed in sect. 3. One adds $N_a$ and $N_b$ dependent terms which produce a minimum for some value of $N_b$, as shown in fig. 4. We find this somewhat unjustified from a microscopic point of view. We have therefore considered another possibility. We start from the hamiltonian (6.5). This hamiltonian conserves separately s, d and p, f bosons. We add to it a term of the type

$$V = -A\hat{D} \cdot \hat{D}, \tag{6.8}$$

where $\hat{D}$ is the dipole operator of (4.7). when $A$ is small, the interaction $V$ can be thought as originating microscopically from a small dipole-dipole interaction.

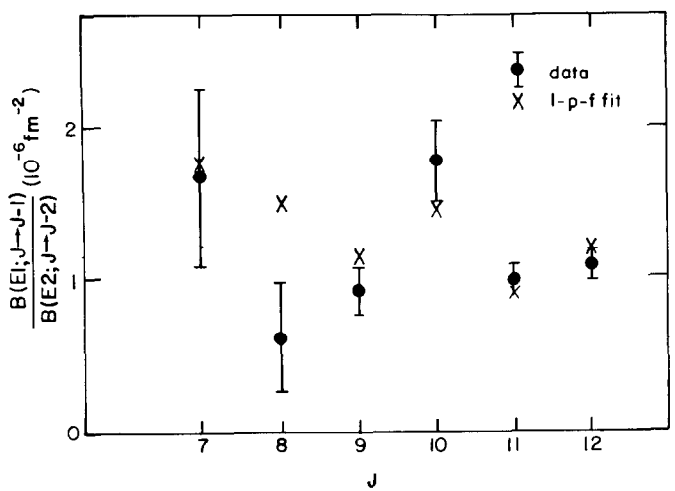


Fig. 9. Measured $^{220}$Ra $R$-ratios and theoretical fits with one p-f boson in the negative-parity band. The E1 parameters in eq. (2.3) are $\alpha_1 = 0.39$, $\beta_1 = 1.70$, $\gamma_1 = -1.55$ (all in units of $10^{-5} e^2\,\mathrm{fm}^2$). The E2 parameters in eq. (6.7) are $\alpha_2' = 1.0$, $\beta_2' = -3.0$ (all in units of $e^2\,\mathrm{fm}^4$).

Alternatively, one could add an interaction

$$V' = -B\hat{\mathcal{P}}^{\dagger} \cdot \hat{\mathcal{P}}, \tag{6.9}$$

where

$$\hat{\mathcal{P}}^{\dagger} = (s^{\dagger} \cdot s^{\dagger}) + (d^{\dagger} \cdot d^{\dagger}) + (p^{\dagger} \cdot p^{\dagger}) + (f^{\dagger} \cdot f^{\dagger}) \tag{6.10}$$

is the pairing operator. The interaction $V'$ would then arise from a pairing interaction. Both interactions (6.8) and (6.9) mix s, d and p, f bosons producing an octupole-dipole deformation (p-f admixtures in the ground state bands).

Fig. 10 shows a fit to the energy levels of $^{218}$Ra obtained by diagonalizing the hamiltonian in the full s, d, p, f space with $N = 6$. This fit is comparable in quality to those shown in fig. 8. The effects of the p-boson on the energies are minor except in the $1^-$ state. However, it appears that even small admixtures of p bosons are enough to obtain well-behaved E1 transitions. Fig. 11 shows a fit for the measured ratio $R$ in $^{218}$Ra. Although specific values for the coefficients $\alpha_1, \beta_1, \gamma_1$ in (2.3) have

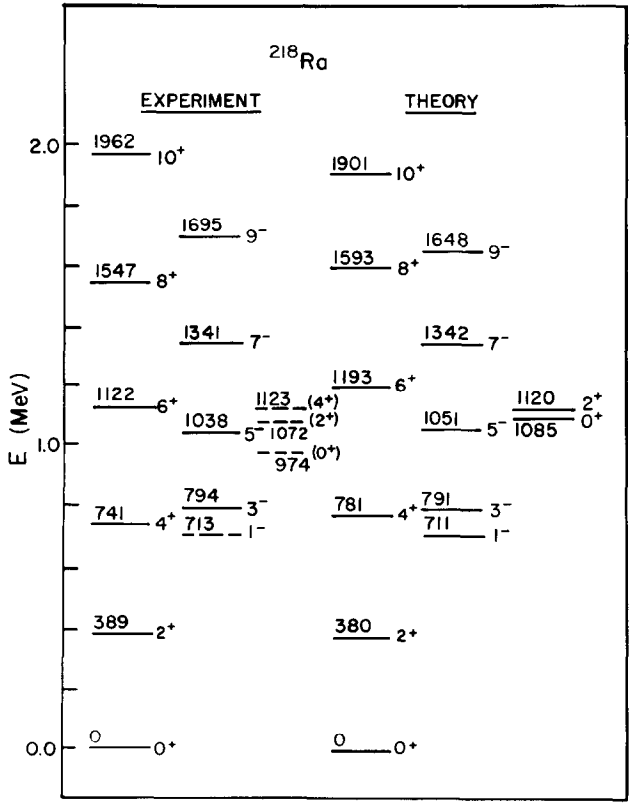


Fig. 10. Measured spectrum of $^{218}$Ra and theoretical fit with many f- and p-bosons. The hamiltonian is given in eqs. (6.5) and (6.8). The parameters are $\varepsilon_d = .55$ MeV, $\kappa = -0.027$ MeV, $\kappa' = -0.147$ MeV, $\varepsilon_p = 1.376$ MeV, $\varepsilon_f = 0.416$ MeV, $\kappa'' = -0.0629$ MeV, $\kappa''' = 0.0119$ MeV, $A = 0.012$ MeV, $\eta = 0$.

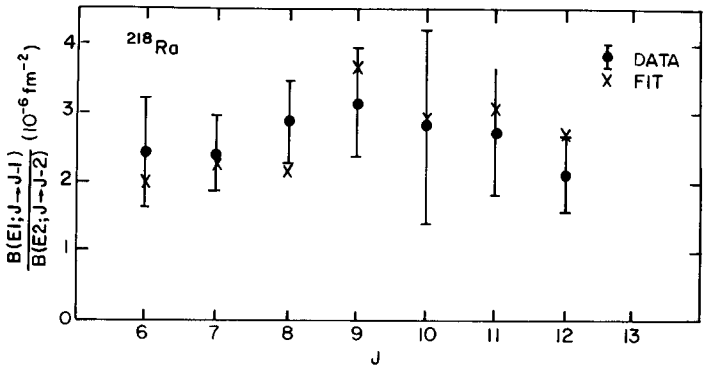


Fig. 11. Measured $^{218}$Ra $R$-ratios and theoretical fits with many f- and p-bosons. The E1 parameters are $\alpha_1 = 0$, $\beta_1 = 2.4$, $\gamma_1 = 0$ (all in units of $10^{-5} e^2\,\mathrm{fm}^2$). The E2 parameters are $\alpha_2' = 1.0$, $\beta_2' = -12.0$ (in units of $e^2\,\mathrm{fm}^4$).

TABLE 4

Expectation values of $\hat{n}_f$ and $\hat{n}_p$ in the yrast band of $^{218}$Ra

| $J$ | $\langle \hat{n}_f \rangle$ | $\langle \hat{n}_p \rangle$ |
|---|---|---|
| 0 | 0.001 | 0.22 |
| 1 | 0.008 | 1.18 |
| 2 | 0.028 | 0.28 |
| 3 | 0.92 | 0.21 |
| 4 | 0.089 | 0.28 |
| 5 | 0.97 | 0.20 |
| 6 | 0.19 | 0.27 |
| 7 | 1.04 | 0.17 |
| 8 | 0.56 | 0.23 |
| 9 | 1.10 | 0.14 |
| 10 | 1.78 | 0.11 |
| 11 | 1.14 | 0.09 |
| 12 | 2.06 | 0.04 |

been used, the results do not depend on balancing contributions of different terms, i.e. no fine-tuning is required. It is also instructive to examine the expectation values of the operators $\hat{n}_f$ and $\hat{n}_p$ in the yrast band of $^{218}$Ra. These are shown in table 4. It is quite interesting that $\langle \hat{n}_f \rangle$ increases with $J$, i.e. the octupole deformation appears to become larger with larger angular momentum.

## 7. Conclusions

In this article we have analyzed a description of collective low-lying negative-parity states in nuclei in terms of interacting bosons, suggesting that p, f bosons be added

to the usual s, d bosons that describe quadrupole degrees of freedom. We have discussed in detail two rotational limits of the corresponding U(16) algebraic model, one, SU(3), generated by quadrupole interactions and the other, O(4), generated by dipole interactions. We have noted that the former is likely to be of greater significance in nuclei than the latter due to the predominance of quadrupole forces between protons and neutrons.

U(16) is a large, complicated algebra and contains other non-rotational dynamic symmetries. Within the realm of groups generated by quadrupole interactions is O(6). This limit does not correspond to stable deformations in all variables and a preliminary examination of the spectra it produces does not immediately suggest applications to known data. A detailed analysis of the situation, however, remains to be done. Another interesting limit, generated by octupole interactions, corresponds to the group O(7). This limit represents *unstable* octupole deformations and also remains to be explored in detail.

Finally, we have analyzed here the experimental data on the Ra isotopes. The dynamic symmetry limits discussed earlier are not relevant here, as these nuclei appear to belong to transitional regions. Numerical analysis of the data indicate that both vibration- and deformation-like interpretations are possible, though the latter, incorporating small ground state octupole correlations, appears to be more natural. Whether examples of rotational dynamic symmetries of the SU(3) or O(4) type can be found remains to be seen; the present experimental situation does not show any evidence of them.

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